THE DISTRIBUTION OF GALOIS ORBITS OF POINTS OF SMALL HEIGHT IN TORIC VARIETIES

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Abstract. In this text, we address the distribution properties of points of small height on proper toric varieties and applications to the related Bogomolov property. We introduce the notion of monocritical toric metrized divisor and we prove that equidistribution occurs for every generic, small sequence with respect to a toric metrized divisor, for every place if and only if the divisor is monocritical. Furthermore, when this is the case, the limit measure is a translate of the natural measure on the compact torus sitting in the principal orbit of the ambient toric variety.

We also study the v-adic modulus distribution of Galois orbits of small points. We characterize, in terms of the given toric semipositive metrized divisor, the cluster measures of v-adic valuations of Galois orbits of generic small sequences.

The Bogomolov property now says that a subvariety of the principal orbit of a proper toric variety that has the same essential minimum than the toric variety with respect to a monocritical toric metrized divisor, must be a translate of a subtorus. We also give several examples, including a non-monocritical divisor for which the Bogomolov property does not hold.

Contents

1. Introduction

The study of the limit distribution of Galois orbits of points of small height was initiated by Szpiro, Ullmo and Zhang in their seminal paper [SUZ97]. They proved the equidistribution of the Galois orbits of sequences of points in an abelian variety over a number field, with Néron-Tate height converging to zero, over the Archimedean places. It was motivated by the Bogomolov conjecture on abelian varieties, and eventually applied in the affirmative answer to this conjecture by Ullmo [Ull98] and Zhang [Zha98], see also [Cin11, Ghi09, Gub07, Yam13] for similar results in the function field case.

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This equidistribution result was widely generalized, in particular to other varieties and other height functions and, with the introduction of Berkovich spaces, to the equidistribution over non-Archimedean places [Bil97, Cha00, FR06, Cha06, BR06, Yua08, BB10, Che11]. We next introduce the necessary background to explain this generalization.

Let $\mathbb{K}$ be a global field, that is, a field which is either a number field or the function field of a regular projective curve over an arbitrary field, and $\mathfrak{M}_{\mathbb{K}}$ its set of places. Let X be a proper algebraic variety over $\mathbb{K}$ of dimension n, and $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}_{\mathbb{K}}})$ a semipositive metrized (Cartier) divisor with D big. Let

$$\mathrm{h}_{\overline{D}}\colon X(\overline{\mathbb{K}}) \longrightarrow \mathbb{R}$$

be the associated height function on the set of algebraic points of X, see §2 for details. It is a generalization of the notion of height of algebraic points considered by Weil, Northcott and others.

The *essential minimum* of X with respect to $\overline{D}$, denoted by $\upmu^{\mathrm{ess}}_{\overline{D}}(X)$, is the smallest possible limit value of the height of a generic net of algebraic points of X. Consequently, we say that a net $(p_l)_{l\in I}$ is $\overline{D}$*-small* if

$$\lim_{l} \mathrm{h}_{\overline{D}}(p_l) = \upmu^{\mathrm{ess}}_{\overline{D}}(X).$$

A fundamental inequality by Zhang [Zha95] shows that the essential minimum can be bounded below in terms of the height and the degree of $\overline{D}$:

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) \geq \frac{\mathrm{h}_{\overline{D}}(X)}{(n+1)\deg_D(D)}. \tag{1.1}$$

We say that $\overline{D}$ is *quasi-canonical* if this lower bound for the essential minimum is an equality (Definition 2.7). Examples of quasi-canonical metrized divisors are given by the canonical metrics on divisors of toric and abelian varieties and, more generally, by the metrics coming from algebraic dynamical systems.

For a place $v \in \mathfrak{M}_{\mathbb{K}}$, we denote by X_v^{an} the v-adic analytification of X. If v is Archimedean, it is a complex space whereas, if v is non-Archimedean, it is a Berkovich space over $\mathbb{C}_v$, the completion of the algebraic closure of the local field $\mathbb{K}_v$. We endow the space of probability measures on X_v^{an} with the weak-$*$ topology with respect to the space of continuous functions on X_v^{an}.

For an algebraic point p of X, we denote by $\mathrm{Gal}(p)_v$ its v-adic Galois orbit, that is, the orbit of p in X_v^{an} under the action of the absolute Galois group of $\mathbb{K}$. We set

$$\mu_{p,v} = \frac{1}{\#\,\mathrm{Gal}(p)_v} \sum_{q\in \mathrm{Gal}(p)_v} \delta_q$$

for the uniform probability measure on $\mathrm{Gal}(p)_v$. We also denote by $\mathrm{c}_1(D, \|\cdot\|_v)^{\wedge n}$ the v-adic Monge-Ampère measure of $\overline{D}$, see for instance [BPS14a, §1.4]. It is a measure on X_v^{an} of total mass $\deg_D(X)$.

The following statement is representative of several equidistribution theorems for Galois orbits of small points in the literature. In this form, it is due to Yuan [Yua08, Theorem 3.1] for number fields and to Gubler [Gub08, Theorem 1.1] for function fields.

Theorem 1.1. *Let X a projective variety over $\mathbb{K}$ of dimension n, and $\overline{D}$ a quasi-canonical semipositive metrized divisor on X with D ample. Let $(p_l)_{l\in I}$ be a generic $\overline{D}$-small net of algebraic points of X. Then, for every $v \in \mathfrak{M}_{\mathbb{K}}$, the net of probability measures $(\mu_{p_l,v})_{l\in I}$ converges to $\frac{1}{\deg_D(X)}\mathrm{c}_1(D, \|\cdot\|_v)^{\wedge n}$, the normalized v-adic Monge-Ampère measure of $\overline{D}$.*

A common feature of this result and its variants and generalizations, is the assumption that the lower bound (1.1) is an equality or, in other words, that the metrized divisor $\overline{D}$ is quasi-canonical. This severely restricts their range of application. Nonetheless, these results do apply to the important case of metrics arising from algebraic dynamical systems and moreover, they have a very strong thesis: not only the Galois orbits of points of small height do converge, but the limit measure is given by the normalized v-adic Monge-Ampère measure.

The motivation of this paper is to start the study what happens when we remove the hypothesis that $\overline{D}$ is quasi-canonical. Some of our typical questions are: is there always an equidistribution phenomenon for Galois orbits of $\overline{D}$-small points? If not, can we give conditions on $\overline{D}$, beyond being quasi-canonical, under which such a phenomenon occurs? When equidistribution occurs, can we describe the limit measure?

We address these questions and some of its continuations in the toric setting. Our approach is based on the techniques developed in the series [BPS14a, BMPS12, BPS14b]. These techniques are well-suited for the study of toric metrics and their associated height functions. In the sequel, we recall the basic constructions.

Let X be a proper toric variety over $\mathbb{K}$ of dimension n, given by a complete fan Σ on a vector space $N_{\mathbb{R}} \simeq \mathbb{R}^n$, and a big toric divisor D on X, given by a virtual support function $\Psi_D\colon N_{\mathbb{R}} \to \mathbb{R}$. This toric divisor also defines an n-dimensional polytope Δ_D in the dual space $M_{\mathbb{R}} := N_{\mathbb{R}}^{\vee}$.

Let $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}_{\mathbb{K}}})$ be a semipositive toric metrized divisor on X. To it we associate an adelic family of concave functions $\psi_{\overline{D},v}\colon N_{\mathbb{R}} \to \mathbb{R}$, $v \in \mathfrak{M}_{\mathbb{K}}$, called the *metric functions* of $\overline{D}$. They satisfy that $|\psi_{\overline{D},v} - \Psi_D|$ is bounded on $N_{\mathbb{R}}$ for all v, and that $\psi_{\overline{D},v} = \Psi_D$ for all v except for a finite number. We also associate to $\overline{D}$ an adelic family of continuous concave functions on the polytope $\vartheta_{\overline{D},v}\colon \Delta_D \to \mathbb{R}$, $v \in \mathfrak{M}_{\mathbb{K}}$, called the *local roof functions* of $\overline{D}$. They verify that $\vartheta_{\overline{D},v}$ is the zero function for all v except for a finite number. The *global roof function* is a concave function $\vartheta_{\overline{D}}\colon \Delta_D \to \mathbb{R}$ defined as a weighted sum of the local roof functions.

The metric functions and the roof functions convey lot of information of the pair $(X, \overline{D})$. For instance, the essential minimum of X with respect to $\overline{D}$ can be computed as the maximum of the global roof function [BPS14b, Theorem 1.1]:

$$\mathrm{\mu}^{\mathrm{ess}}_{\overline{D}}(X) = \max_{x\in\Delta_D} \vartheta_{\overline{D}}(x). \tag{1.2}$$

In the toric setting, the condition that the metrized divisor $\overline{D}$ is quasi-canonical is very restrictive, since it is equivalent to the condition that its global roof function is constant (Proposition 5.3). Thus, the only toric metrics to which the equidistribution theorem 1.1 applies are those whose global roof function is constant.

To identify the toric metrics having good equidistribution properties, we introduce the notion of *monocritical* toric metrized divisor. The semipositive toric metrized divisor $\overline{D}$ is monocritical if a certain function on a space of measures attains its infimum at a unique measure (Definition 4.14). When this is the case, the minimizing measure determines a point

$$\boldsymbol{u} = (u_v)_{v\in\mathfrak{M}_{\mathbb{K}}} \in \bigoplus_{v\in\mathfrak{M}_{\mathbb{K}}} N_{\mathbb{R}}$$

with $\sum_v n_v u_v = 0$, called the *critical point* of $\overline{D}$.

The condition for $\overline{D}$ of being monocritical can be characterized in terms of its global roof function: given a point $x_{\max} \in \Delta_D$ maximizing $\vartheta_{\overline{D}}$, the sup-differential $\partial\vartheta_{\overline{D}}(x_{\max})$ is a convex subset of $N_{\mathbb{R}}$ containing the point 0. Then $\overline{D}$ is monocritical if and only if 0 is a vertex of this convex subset and, when this is the case, the critical

point of $\overline{D}$ can be computed from the sup-differential of the local roof functions at $x_{\max}$ (Proposition 4.16).

Let $\mathbb{T} \simeq \mathbb{G}_{\mathrm{m},\mathbb{K}}^n$ be the torus of X, which can be identified with X_0, the principal open subset of X. For each $v \in \mathfrak{M}_{\mathbb{K}}$, we denote by $\mathbb{S}_v$ the compact subtorus of $\mathbb{T}_v^{\mathrm{an}}$. To a monocritical toric metrized divisor $\overline{D}$ with critical point $\boldsymbol{u} \in \bigoplus_{v\in\mathfrak{M}_{\mathbb{K}}} N_{\mathbb{R}}$, we associate a probability measure $\lambda_{\mathbb{S}_v,u_v}$ on X_v^{an} (Definition 5.1). When v is Archimedean, it is the uniform measure on a translate of $\mathbb{S}_v \simeq (S^1)^n$ whereas, when v is non-Archimedean, it is the Dirac measure at a translate of the Gauss point of $\mathbb{T}_v^{\mathrm{an}}$.

The following is the main result of this paper (Theorem 5.2).

Theorem 1.2. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a semipositive toric metrized divisor on X with D big. Then $\overline{D}$ is monocritical if and only if for every place $v \in \mathfrak{M}_{\mathbb{K}}$ and every generic $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X_0, the net of probability measures $(\mu_{p_l,v})_{l\in I}$ on X_v^{an} converges.*

When this is the case, the limit measure agrees with $\lambda_{\mathbb{S}_v,u_v}$, where $u_v \in N_{\mathbb{R}}$ is the v-adic component of the critical point of $\overline{D}$.

Quasi-canonical toric metrized divisors are monocritical, and Theorem 1.2 reduces to Theorem 1.1 in this case. Our result produces a wealth of new examples of metric satisfying the equidistribution property, that were not covered by the previous results. In particular, this is the case for toric divisors over a number field $\mathbb{K}$ with positive smooth metrics at the Archimedean places and canonical metrics at the non-Archimedean ones (Theorem 6.4). Here we state a simplified version for the case when $\mathbb{K} = \mathbb{Q}$.

Corollary 1.3. *Let X be a proper toric variety over $\mathbb{Q}$ and $\overline{D}$ a semipositive toric metrized $\mathbb{R}$-divisor with D big. We assume that the v-adic metric of $\overline{D}$ is, when v is the Archimedean place, smooth and positive and, when v is non-Archimedean, equal to the v-adic canonical metric of D. Then, for every generic $\overline{D}$-small sequence $(p_l)_{l\geq 1}$ of algebraic points of X_0 and every place $v \in \mathfrak{M}_{\mathbb{Q}}$, the sequence $(\mu_{p_l,v})_{l\geq 1}$ on X_v^{an} converges to the probability measure $\lambda_{\mathbb{S}_v,0}$.*

This corollary covers many typical examples of metrics on toric varieties like weighted projective spaces and toric bundles, see § 6.2. For instance, let $X = \mathbb{P}^1_{\mathbb{Q}}$ and $\overline{D}$ the divisor of the point at infinity equipped with the Fubini-Study metric at the Archimedean place and the canonical metric at the non-Archimedean places. Its essential minimum is

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) = \frac{\log(2)}{2}$$

and, for every generic sequence of algebraic points of $\mathbb{P}^1_{\mathbb{Q}}$ with height converging to this quantity, its ∞-adic Galois orbits converge to the Haar probability measure on S^1, the unit circle of the Riemann sphere (Example 6.5). This an example where equidistribution does occur, but the limit measure is *not* given by the v-adic Monge-Ampère measure as in Theorem 1.1.

In the other extreme, classical examples of translates of subtori with the canonical metric can behave badly with respect to equidistribution. For instance let X be the line of $\mathbb{P}^2_{\mathbb{Q}}$ of equation $2z_1 - z_2 = 0$ and $\overline{D}$ the metrized divisor on X given by the restriction of the canonical metrized divisor at infinity of $\mathbb{P}^2_{\mathbb{Q}}$. As explained in Example 6.1, Theorem 1.2 implies that $\overline{D}$ does not satisfy the equidistribution property in the sense of Definition 2.9.

We also study the modulus distribution of the v-adic Galois orbits of $\overline{D}$-small nets of algebraic points in the general, non necessarily monocritical, toric case.

There is a valuation map $\mathrm{val}_v\colon \mathbb{T}_v^{\mathrm{an}} \to N_{\mathbb{R}}$, defined, in any given splitting, by

$$\mathrm{val}_v(x_1,\dots,x_n) = (-\log|x_1|_v,\dots,-\log|x_n|_v).$$

Hence, for an algebraic point p of $X_0 = \mathbb{T}$, the direct image measure

$$\nu_{p,v} \coloneqq (\mathrm{val}_v)_*\mu_{p,v}$$

is a probability measure on $N_{\mathbb{R}}$ that gives the modulus distribution of its v-adic Galois orbit.

To each semipositive toric metrized divisor $\overline{D}$ with D big, we associate an adelic family of nonempty subsets of $N_{\mathbb{R}}$

$$(B_v, F_v)_{v\in\mathfrak{M}_{\mathbb{K}}}, \tag{1.3}$$

with $B_v \subset F_v$ (Notation 4.2). We endow the space of probability measures on $N_{\mathbb{R}}$ with the weak-$*$ topology with respect to the bounded continuous functions on $N_{\mathbb{R}}$. For a probability measure ν on $N_{\mathbb{R}}$, we denote by $\mathrm{supp}(\nu) \subset N_{\mathbb{R}}$ its support and, if ν has finite first moment, we denote by $\mathrm{E}[\nu]$ its expected value.

The next result characterizes the limit behavior of the modulus distribution for $\overline{D}$-small nets (Theorem 4.3 and Corollary 4.12).

Theorem 1.4. *Let X be a proper toric variety over $\mathbb{K}$, $\overline{D}$ a semipositive toric metrized divisor on X with D big, and $v \in \mathfrak{M}_{\mathbb{K}}$. For every $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points in X_0, the net of probability measures $(\nu_{p_l,v})_{l\in I}$ has at least one cluster point. Every such cluster point is a measure ν_v with finite first moment that satisfies*

$$\mathrm{supp}(\nu_v) \subset F_v \quad \textit{and} \quad \mathrm{E}[\nu_v] \in B_v. \tag{1.4}$$

Conversely, for every probability measure ν_v on $N_{\mathbb{R}}$ which has finite first moment and satisfies (1.4)*, there is a $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X_0 such that ν_v is the limit of the net $(\nu_{p_l,v})_{l\in I}$.*

In the situation of Theorem 1.4, when F_v consist of only one point u_v, the net $(\nu_{p_l,v})_{l\in I}$ converges to the measure δ_{u_v}. In this case we say that $\overline{D}$ satisfies the *modulus concentration property* at the place v.

In fact, $\overline{D}$ is monocritical if and only if for every place v, the set F_v consists of only one point. Hence, our results imply that $\overline{D}$ satisfies the equidistribution property at every place if and only if it satisfies the modulus concentration property at every place. Observe also that, in contrast with Theorem 1.2, the net of algebraic points in Theorem 1.4 does not need to be generic.

In the absence of modulus concentration, there is a wealth of limit measures of v-adic Galois orbits of $\overline{D}$-small nets of algebraic points. For instance, consider the projective line over a number field $\mathbb{K}$ and any adelic set $\boldsymbol{E} = (E_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$ of global capacity 1, whose associated equilibrium measures are compatible with the collection of sets in (1.3) (see Theorem 7.2 for the precise condition). Using Rumely's Fekete-Szegő theorem [Rum02], we show that, for all v, the equilibrium measure of E_v can be realized as the limit measure of a sequence of v-adic Galois orbits of $\overline{D}$-small points (Theorem 7.2).

As we already mentioned, the original motivation in [SUZ97] to search for equidistribution results of Galois orbits of small points was to prove the Bogomolov conjecture. The Bogomolov conjecture for toric varieties can be stated as follows: let X be a toric variety over $\mathbb{K}$ and $\overline{D}^{\mathrm{can}}$ an ample toric divisor on X equipped with the canonical metric. Let $V \subset X_{0,\overline{\mathbb{K}}}$ be a subvariety which is not a translate of a subtorus by a torsion point. Then there exists $\varepsilon > 0$ such that the subset of algebraic points of V of canonical height bounded above by ε, is not dense in V.

Equivalently, if $V \subset X_{0,\overline{\mathbb{K}}}$ is a subvariety with $\upmu^{\mathrm{ess}}_{\overline{D}^{\mathrm{can}}}(V) = 0$, then V is a translate of a subtorus by a torsion point. It is the toric counterpart of the Bogomolov conjecture for abelian varieties proved by Ullmo and Zhang.

This conjecture was proved by Zhang [Zha95] for number fields, and later Bilu gave a proof using his equidistribution theorem [Bil97]. Here we extend this result to an arbitrary monocritical metric on a toric variety over a number field (Theorem 5.12).

Theorem 1.5. *Let X be a proper toric variety over a number field $\mathbb{K}$ and $\overline{D}$ a monocritical toric metrized divisor on X with critical point $\boldsymbol{u} = (u_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$. Let V be a subvariety of $X_{0,\overline{\mathbb{K}}}$ with*

$$\upmu^{\mathrm{ess}}_{\overline{D}}(V) = \upmu^{\mathrm{ess}}_{\overline{D}}(X).$$

Then V is a translate of a subtorus. Furthermore, if $u_v \in \mathrm{val}_v(\mathbb{T}(\mathbb{K})) \otimes \mathbb{Q}$ for all v, then V is the translate of a subtorus by an algebraic point p of X_0 with $\mathrm{h}_{\overline{D}}(p) = \upmu^{\mathrm{ess}}_{\overline{D}}(X)$.

A subvariety of $X_{0,\overline{\mathbb{K}}}$ with

$$\upmu^{\mathrm{ess}}_{\overline{D}}(V) = \upmu^{\mathrm{ess}}_{\overline{D}}(X)$$

is called a $\overline{D}$-special subvariety. We say that a given toric metrized divisor $\overline{D}$ satisfies the *Bogomolov property*[1] if every $\overline{D}$-special subvariety is a translate of a subtorus (Definition 5.11). This property is intimately related with the equidistribution property. Indeed, we give an example of a metrized divisor $\overline{D}$ on $\mathbb{P}^2_{\mathbb{Q}}$, such that the line of equation $z_0 + z_1 + z_2 = 0$ is $\overline{D}$-special (Example 6.6). This line is certainly not a translate of a subtorus, and so $\overline{D}$ does not satisfy the Bogomolov property. This metrized divisor is a variant of the one in Example 6.1, and does not verify modulus concentration nor equidistribution for any place of $\mathbb{Q}$.

We comment briefly on the techniques of proof. To prove Theorem 1.4, for each place v we construct an upper-semicontinuous concave functional Φ_v on the space of probability measures with finite first moment on $N_{\mathbb{R}}$. Using the toric dictionary from [BPS14a, BMPS12, BPS14b], we reduce the study of the modulus distribution of v-adic Galois orbits to the optimization of this functional, which we treat by applying Prokorov's compactness theorem.

Once we know that a monocritical divisor $\overline{D}$ satisfies modulus concentration at every place, we can construct another toric metric on D that is quasi-canonical and such that the $\overline{D}$-small points are also small with respect to this new metric. With this trick, the toric equidistribution theorem 1.2 is derived from Theorem 1.1.

The Bogomolov property for monocritical metrics (Theorem 1.5) follows from Theorem 5.7, a variant of the toric equidistribution theorem for $\overline{D}$-small nets that are not necessarily generic but only *strict*, in the sense that they eventually avoid any fixed translate of subtorus.

These results arise several interesting questions. For instance: is it possible that a given semipositive toric metrized divisor $\overline{D}$ satisfies the equidistribution property at one place and not at another? We study this for the projective line showing that, under a natural rationality hypothesis, the equidistribution property holds at a given place if and only if it holds at every place (Proposition 7.5). However, this conclusion is not true without this rationality hypothesis (Remark 7.7) and we have neither settled this question for the projective line in full generality, nor treated toric varieties of higher dimension.

[1] not to be confused with the property (B) introduced by Bombieri and Zannier, and studied by Amoroso, David and other authors.

It would also be interesting to see if the converse of Theorem 1.5 holds: Let X be proper toric variety with $\dim X \geq 2$. Given a semipositive toric metrized divisor $\overline{D}$ on X, with D big satisfying the Bogomolov property, is $\overline{D}$ necessarily monocritical? In Proposition 6.7 we show that this is true in a very particular case. Extending this to the general case would reinforce the link between the equidistribution and the Bogomolov properties.

The results of this paper also inspire questions for general varieties and metrized divisors. For instance, from Corollary 1.3, it is plausible to conjecture that a toric divisor equipped with a positive smooth, but not necessarily toric, Archimedean metric and canonical non-Archimedean metrics, does satisfy the equidistribution property. A puzzling question is that of computing the essential minimum, with a formula generalizing (1.2) to the general, non-toric, case. Even more challenging seems the problem of generalizing the crucial notion of monocritical metrized divisor.

Several of the results presented in this introduction hold in greater generality and their thesis are stronger. We refer to the body of the paper for these versions. The structure of the paper is as follows. In §2 we give the preliminaries on Galois orbits and height of points. In §3 we introduce the upper semi-continuous concave functional Φ_v and study its properties. In §4 we study the modulus distribution of v-adic Galois orbits of $\overline{D}$-small nets of points in toric varieties. In §5 we prove the toric equidistribution theorem 1.2 and its variants, together with the Bogomolov property for monocritical toric metrized divisors. In §6 we give examples illustrating a number of phenomena, including a non-monocritical toric metrized divisor not verifying the Bogomolov property. Finally, in §7 we use potential theory to study the limit measures that appear in the absence of modulus concentration.

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2. Galois orbits, height of points and essential minimum

By a *global field* $\mathbb{K}$ we mean either a number field or the function field of a regular projective curve over an arbitrary field, equipped with a certain set of places, denoted by $\mathfrak{M}_{\mathbb{K}}$. Each place $v \in \mathfrak{M}_{\mathbb{K}}$ is a pair consisting of an absolute value $|\cdot|_v$ on $\mathbb{K}$ and a positive weight $n_v \in \mathbb{Q}_{>0}$, defined as follows.

The places of the field of rational numbers $\mathbb{Q}$ consist of the Archimedean and the p-adic absolutes values, normalized in the standard way, and with all weights equal to 1. For the function field $\mathrm{K}(C)$ of a regular projective curve C over a field k, the set of places is indexed by the closed points of C. For each closed point $v_0 \in C$, we consider the absolute value and weight given, for $\alpha \in \mathrm{K}(C)^{\times}$, by

$$|\alpha|_{v_0} = c_k^{-\mathrm{ord}_{v_0}(\alpha)} \quad \text{and} \quad n_{v_0} = [k(v_0) : k],$$

with $c_k = \#k$ if the base field k is finite and $c_k = \mathrm{e}$ otherwise, and where $\mathrm{ord}_{v_0}(\alpha)$ denotes the order of α in the discrete valuation ring $\mathcal{O}_{C,v_0}$.

Let $\mathbb{K}_0$ denote either $\mathbb{Q}$ or $\mathrm{K}(C)$. In the general case when $\mathbb{K}$ is a finite extension of $\mathbb{K}_0$, the set of places of $\mathbb{K}$ is formed by the pairs $v = (|\cdot|_v, n_v)$ where $|\cdot|_v$ is an absolute value on $\mathbb{K}$ extending an absolute value $|\cdot|_{v_0}$ with $v_0 \in \mathfrak{M}_{\mathbb{K}_0}$ and

$$n_v = \frac{[\mathbb{K}_v : \mathbb{K}_{0,v_0}]}{[\mathbb{K} : \mathbb{K}_0]} n_{v_0}, \tag{2.1}$$

where $\mathbb{K}_v$ denotes the completion of $\mathbb{K}$ with respect to $|\cdot|_v$, and similarly for $\mathbb{K}_{0,v_0}$. This set of places satisfies the following basic properties.

Proposition 2.1. *Let $\mathbb{K}_0$ denote either $\mathbb{Q}$ or $\mathrm{K}(C)$, the function field of a regular projective curve C over a field k. Let $\mathbb{K}$ be a finite extension of $\mathbb{K}_0$ and $\mathfrak{M}_\mathbb{K}$ the associated set of places as above. Then*

(1) *for all $v_0 \in \mathfrak{M}_{\mathbb{K}_0}$, we have $\sum_{v|v_0} n_v = n_{v_0}$;*
(2) *for all $\alpha \in \mathbb{K}^\times$, we have $\sum_{v\in\mathfrak{M}_\mathbb{K}} n_v \log|\alpha|_v = 0$ (product formula).*

Proof. These properties are classical, see for instance [AW45, Theorems 2 and 3].

In the function field case there is a subtlety, due to the fact that a given field may have different structures of global field depending on the choice of base curve.

Let C be a regular projective curve over k and $K(C) \hookrightarrow \mathbb{K}$ a finite extension of fields. Then there is a regular projective curve B over k and a finite morphism $\pi\colon B \to C$ such that $\mathbb{K} \simeq K(B)$ and the previous extension can be identified with $\pi^*\colon K(C) \hookrightarrow K(B)$, see for instance [Liu02, Proposition 7.3.13 and Lemma 7.3.10].

We could give to $\mathbb{K}$ the structure of global field defined directly by the curve B, but the obtained absolute values of $\mathbb{K}$ would not be extensions of those of $\mathbb{K}_0$. To remedy this, we renormalize these absolute values of $\mathbb{K}$ and, to preserve the product formula, we also change the weights.

From the valuative criterium of properness, for each closed point $v_0 \in C$, the absolute values of $\mathbb{K}$ extending $|\cdot|_{v_0}$ are in bijection with the closed points of the fiber above v_0. Moreover, since the map π is finite, for each closed point $v \in \pi^{-1}(v_0)$, the ring $\mathcal{O}_{B,v}$ is a finite module over $\mathcal{O}_{C,v_0}$. It follows from [Bou85, Chapitre 6, Proposition 2 in §8.2 and Theorem 2 in §8.5] that the absolute value and weight corresponding to v are given, for $\alpha \in \mathrm{K}(B)^\times$, by

$$|\alpha|_v = c_k^{-\frac{\mathrm{ord}_v(\alpha)}{e_{v/v_0}}}, \quad n_v = \frac{e_{v/v_0}[k(v):k]}{[\mathrm{K}(B):\mathrm{K}(C)]}, \tag{2.2}$$

with e_{v/v_0} the ramification index of v over v_0. The same results in *loc. cit.* give the formula in (1).

For the product formula in (2), we obtain from (2.2) that

$$\sum_v n_v \log|\alpha|_v = -\log(c_k)\sum_v \frac{[k(v):k]\mathrm{ord}_v(\alpha)}{[\mathrm{K}(B):\mathrm{K}(C)]} = \frac{-\log(c_k)}{[\mathrm{K}(B):\mathrm{K}(C)]}\deg(\mathrm{div}(\alpha)) = 0,$$

because the degree of a principal divisor on B is zero, which concludes the proof. $\square$

For $v \in \mathfrak{M}_\mathbb{K}$, we choose an algebraic closure $\mathbb{K}_v \subset \overline{\mathbb{K}}_v$ of $\mathbb{K}_v$. The absolute value $|\cdot|_v$ on $\mathbb{K}_v$ has a unique extension to $\overline{\mathbb{K}}_v$. We denote by $\mathbb{C}_v$ the completion of $\overline{\mathbb{K}}_v$ with respect to this extended absolute value. We also choose an algebraic closure $\overline{\mathbb{K}}$ of $\mathbb{K}$ and an embedding $\jmath_v\colon \overline{\mathbb{K}} \to \mathbb{C}_v$.

Let X be a variety over $\mathbb{K}$, that is, a reduced and irreducible separated scheme of finite type over $\mathbb{K}$. The elements of $X(\overline{\mathbb{K}})$ are called the *algebraic points* of X. For $p \in X(\overline{\mathbb{K}})$, its *Galois orbit* is $\mathrm{Gal}(p) := \mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K})\cdot p \subset X(\overline{\mathbb{K}})$, that is, the orbit of p under the action of the absolute Galois group of $\mathbb{K}$.

For $v \in \mathfrak{M}_\mathbb{K}$, we denote by $X^{\mathrm{an}}_{\mathbb{K}_v}$ the v-adic analytifications of X over $\mathbb{K}_v$ and by X^{an}_v the v-adic analytifications of X over $\mathbb{C}_v$. If v is Archimedean, they both coincide with a complex space ($X_{\mathbb{K}_v}$ is equipped with an anti-linear involution if $\mathbb{K}_v \simeq \mathbb{R}$). If v is non-Archimedean, they are Berkovich spaces over $\mathbb{K}_v$ and $\mathbb{C}_v$, respectively. These spaces are related by ([Ber90, Corollary 1.3.6])

$$X^{\mathrm{an}}_{\mathbb{K}_v} = X^{\mathrm{an}}_v / \mathrm{Gal}(\overline{\mathbb{K}}_v/\mathbb{K}_v).$$

We denote by

$$\pi_v\colon X^{\mathrm{an}}_v \to X^{\mathrm{an}}_{\mathbb{K}_v} \tag{2.3}$$

the projection.

There is a map

$$X(\mathbb{C}_v) \longrightarrow X_v^{\mathrm{an}}.$$

Using the chosen inclusion $\jmath_v\colon \overline{\mathbb{K}} \hookrightarrow \mathbb{C}_v$, we obtain a map $X(\overline{\mathbb{K}}) \hookrightarrow X(\mathbb{C}_v)$ and, by composition the previous map, an inclusion

$$\iota_v\colon X(\overline{\mathbb{K}}) \hookrightarrow X_v^{\mathrm{an}}.$$

The *v-adic Galois orbit* of an algebraic point $p \in X(\overline{\mathbb{K}})$, denoted by $\mathrm{Gal}(p)_v$, is defined as the image of $\mathrm{Gal}(\overline{\mathbb{K}}/\mathbb{K}) \cdot p$ under ι_v. It is a finite subset which does not depend on the choice of the inclusion $\jmath_v$. We also denote by $\mu_{p,v}$ the uniform discrete probability measure on X_v^{an} supported on $\mathrm{Gal}(p)_v$, that is

$$\mu_{p,v} = \frac{1}{\#\,\mathrm{Gal}(p)_v} \sum_{q\in \mathrm{Gal}(p)_v} \delta_q, \tag{2.4}$$

where δ_q is the Dirac measure at the point $q \in X_v^{\mathrm{an}}$. Hence, for a continuous function $f\colon X_v^{\mathrm{an}} \to \mathbb{R}$,

$$\int f \,\mathrm{d}\mu_{p,v} = \frac{1}{\#\,\mathrm{Gal}(p)_v} \sum_{q\in \mathrm{Gal}(p)_v} f(q).$$

An $\mathbb{R}$-*divisor* on X is a linear combination of Cartier divisors on X with real coefficients. A *metrized* $\mathbb{R}$-divisor $\overline{D}$ on X is an $\mathbb{R}$-divisor D on X equipped with a quasi-algebraic family of v-adic metrics $(\|\cdot\|_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$, see [BMPS12, §3] for details. In *loc. cit.*, for each $v \in \mathfrak{M}_{\mathbb{K}}$ the metric $\|\cdot\|_v$ is defined over the analytic space $X_{\mathbb{K}_v}^{\mathrm{an}}$. Note that this space was denoted "X_v^{an}" in *loc. cit.* but since we will study equidistribution problems of Galois orbits of points that are defined over varying extensions of $\mathbb{K}$ of arbitrary large degree it is more convenient to work on the space X_v^{an} instead that in the space $X_{\mathbb{K}_v}^{\mathrm{an}}$. Hence we have changed the notation accordingly. With this point of view, every object on $X_{\mathbb{K}_v}^{\mathrm{an}}$ will be seen as an object on X_v^{an} by taking its inverse image under the projection π_v. For instance let $\overline{D}$ be a metrized $\mathbb{R}$-divisor on X and s a rational $\mathbb{R}$-section of D [BMPS12, §3]. In *loc. cit.*, the v-adic metric $\|\cdot\|_v$ is described by a continuous function $\|s\|_v\colon X_{\mathbb{K}_v}^{\mathrm{an}} \setminus |\operatorname{div}(s)| \to \mathbb{R}_{>0}$. In the current paper we denote by $\|s\|_v$ the function on $X_v^{\mathrm{an}} \setminus |\operatorname{div}(s)|$ given by the composition

$$\|s(p)\|_v = \|s(\pi_v(p))\|_v.$$

Clearly this function is invariant under the action of $\mathrm{Gal}(\overline{\mathbb{K}}_v, \mathbb{K}_v)$.

To a metrized $\mathbb{R}$-divisor $\overline{D}$ on X we can associate a height function

$$\mathrm{h}_{\overline{D}}\colon X(\overline{\mathbb{K}}) \longrightarrow \mathbb{R}$$

as follows.

Given $p \in X(\overline{\mathbb{K}})$, choose a rational $\mathbb{R}$-section s of D such that $p \notin |\operatorname{div}(s)|$. Choose a finite extension $\mathbb{F}$ of $\mathbb{K}$ such that $p \in X(\mathbb{F})$. For each $w \in \mathfrak{M}_{\mathbb{F}}$ over a place $v \in \mathfrak{M}_{\mathbb{K}}$, we can choose an embedding $\sigma_w\colon \mathbb{F} \hookrightarrow \mathbb{C}_v$ such that the restriction of the absolute value $|\cdot|_v$ of $\mathbb{C}_v$ agrees with $|\cdot|_w$. We denote also by σ_w the induced map $X(\mathbb{F}) \to X_v^{\mathrm{an}}$.

Definition 2.2. Let X be a variety over $\mathbb{K}$, $\overline{D}$ a metrized $\mathbb{R}$-divisor on X, and $p \in X(\overline{\mathbb{K}})$. With the above notation, the *height* of p with respect to $\overline{D}$ is defined as

$$\mathrm{h}_{\overline{D}}(p) = -\sum_{w\in\mathfrak{M}_{\mathbb{F}}} n_w \log \|s \circ \sigma_w(p)\|_v.$$

The height is independent of the choice of the rational $\mathbb{R}$-section s, the extension $\mathbb{F}$ and the embeddings σ_w.

Instead of choosing a finite extension where the point p is defined, it is possible to express the height of an algebraic point in terms of its Galois orbit.

Proposition 2.3. *With the previous hypothesis and notation, the height of p with respect to $\overline{D}$ is given by*

$$\mathrm{h}_{\overline{D}}(p) = -\sum_{v\in\mathfrak{M}_{\mathbb{K}}} \frac{n_v}{\#\operatorname{Gal}(p)_v} \sum_{q\in\operatorname{Gal}(p)_v} \log\|s(q)\|_v.$$

Proof. Choose a finite normal extension $\mathbb{F}\subset\overline{\mathbb{K}}$ of $\mathbb{K}$ such that $p\in X(\mathbb{F})$. For each $v\in\mathfrak{M}_{\mathbb{K}}$ we denote $\mathfrak{M}_{\mathbb{F},v}$ the set of places of $\mathbb{F}$ above v.

Write $G=\operatorname{Gal}(\mathbb{F},\mathbb{K})$ and let $\mathbb{F}^G$ be the fixed field. Then $\mathbb{F}/\mathbb{F}^G$ is a Galois extension with Galois group G and $\mathbb{F}^G/\mathbb{K}$ is purely inseparable. Hence, for $v\in\mathfrak{M}_{\mathbb{K}}$,

$$\frac{[\mathbb{F}_w:\mathbb{K}_v]}{[\mathbb{F}:\mathbb{K}]} = \frac{[\mathbb{F}_w:(\mathbb{F}^G)_v]}{[\mathbb{F}:\mathbb{F}^G]} = \frac{1}{\#\mathfrak{M}_{\mathbb{F},v}}.$$

Then, from the definition of the height of p in Definition 2.2 and Proposition 2.1(1), it follows that

$$\begin{aligned}\mathrm{h}_{\overline{D}}(p) &= -\sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \sum_{w\mid v} \frac{[\mathbb{F}_w:\mathbb{K}_v]}{[\mathbb{F}:\mathbb{K}]} \log\|s\circ\sigma_w(p)\|_v \\ &= -\sum_{v\in\mathfrak{M}_{\mathbb{K}}} \frac{n_v}{\#\mathfrak{M}_{\mathbb{F},v}} \sum_{w\mid v} \log\|s\circ\sigma_w(p)\|_v. \qquad (2.5)\end{aligned}$$

The group G acts on $\mathfrak{M}_{\mathbb{F},v}$ and, since p is defined over $\mathbb{F}$, also on $\operatorname{Gal}(p)_v$. Both actions are transitive. Therefore, choosing $w_0\in\mathfrak{M}_{\mathbb{F},v}$,

$$\begin{aligned}\frac{1}{\#\mathfrak{M}_{\mathbb{F},v}} \sum_{w\mid v} \log\|s\circ\sigma_w(p)\|_v &= \frac{1}{\#G}\sum_{g\in G} \log\|s\circ\sigma_{w_0}(g(p))\|_v \\ &= \frac{1}{\#\operatorname{Gal}(p)_v} \sum_{q\in\operatorname{Gal}(p)_v} \log\|s(q)\|_v.\end{aligned}$$

The statement follows from this together with (2.5). □

The *essential minimum* of X with respect to $\overline{D}$ is defined as

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) = \sup_{\substack{Y\subsetneq X\\ Y \text{ closed}}} \inf_{p\in(X\setminus Y)(\overline{\mathbb{K}})} \mathrm{h}_{\overline{D}}(p). \qquad (2.6)$$

Roughly speaking, the essential minimum is the generic infimum of the height function.

Definition 2.4. Let X be a variety over $\mathbb{K}$ and $\overline{D}$ a metrized $\mathbb{R}$-divisor on X. A net $(p_l)_{l\in I}$ of algebraic points of X is *$\overline{D}$-small* if

$$\lim_l \mathrm{h}_{\overline{D}}(p_l) = \upmu^{\mathrm{ess}}_{\overline{D}}(X).$$

The net $(p_l)_{l\in I}$ is *generic* if, for every closed subset $Y\subsetneq X$, there is $l_0\in I$ such that $p_l\notin Y(\overline{\mathbb{K}})$ for $l\geq l_0$.

Proposition 2.5. *Given a variety X over $\mathbb{K}$ and $\overline{D}$ a metrized $\mathbb{R}$-divisor on X, there exists a generic $\overline{D}$-small net of algebraic points of X. Moreover, every generic net $(p_l)_{l\geq 1}$ of algebraic points of X satisfies*

$$\liminf_l \mathrm{h}_{\overline{D}}(p_l) \geq \upmu^{\mathrm{ess}}_{\overline{D}}(X).$$

Proof. The second statement is clear from the definition of the essential minimum.

For the first statement, let I be the set of hypersurfaces of X, ordered by inclusion. This is a directed set. For each $Y \in I$, denote by $c(Y)$ its number of irreducible components and choose a point $p_Y \in (X \setminus Y)(\overline{\mathbb{K}})$ with

$$\mathrm{h}_{\overline{D}}(p_Y) \leq \upmu^{\mathrm{ess}}_{\overline{D}}(X) + \frac{1}{c(Y)}.$$

Clearly, the net $(p_Y)_{Y\in I}$ is generic and $\overline{D}$-small. □

Remark 2.6. When $\mathbb{K}$ is a number field, the collection of subvarieties of X is countable. Using this fact, we can strengthen Proposition 2.5 to show the existence of a generic $\overline{D}$-small *sequence* of algebraic points.

Suppose now that the variety X is proper over $\mathbb{K}$ and of dimension n. A metrized $\mathbb{R}$-divisor $\overline{D}$ on X is *semipositive* if it can be written as

$$\overline{D} = \sum_{i=1}^{r} \alpha_i \overline{D}_i$$

with $\overline{D}_i$ a semipositive metrized divisor and $\alpha_i \in \mathbb{R}_{\geq 0}$, $i = 1, \dots, r$. Recall that $\overline{D}_i$ is semipositive if each of its v-adic metrics is a uniform limit of a sequence of semipositive smooth (respectively, algebraic) metrics in the Archimedean (respectively, non-Archimedean) case.

Given a semipositive metrized $\mathbb{R}$-divisor $\overline{D}$, we can extend the notion of height of points to subvarieties of higher dimension. In particular, the height of X, denoted by $\mathrm{h}_{\overline{D}}(X)$, is defined. Moreover, for each $v \in \mathfrak{M}_{\mathbb{K}}$ we can consider the associated v-adic Monge-Ampère measure, denoted by $\mathrm{c}_1(D, \|\cdot\|_v)^{\wedge n}$. It is a measure on X_v^{an} of total mass $\deg_D(X)$, see for instance [BPS14a, §1.4] for the case when D is a divisor. The v-adic Monge-Ampère measure of an $\mathbb{R}$-divisor is defined from that of divisors by polarization and multilinearity.

A theorem of Zhang shows that, when $\mathbb{K}$ is a number field, D is an ample divisor and $\overline{D}$ is semipositive, the essential minimum can be bounded below in terms of the height of X and the degree of D [Zha95, Theorem 5.2]:

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) \geq \frac{\mathrm{h}_{\overline{D}}(X)}{(n+1)\deg_D(D)}. \tag{2.7}$$

This inequality can be generalized to global fields and semiample big divisors, see for instance [Gub08, Proposition 5.10].

In some cases, the inequality (2.7) is an equality. For instance, this happens for the canonical metric on divisors of toric and abelian varieties, and for the canonical metrics coming from dynamical systems. This motivates the following definition.

Definition 2.7. Let X be a proper variety over $\mathbb{K}$ of dimension n, and $\overline{D}$ a semipositive metrized $\mathbb{R}$-divisor on X with D big. Then $\overline{D}$ is *quasi-canonical* if

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) = \frac{\mathrm{h}_{\overline{D}}(X)}{(n+1)\deg_D(X)}.$$

In other words, quasi-canonical metrized $\mathbb{R}$-divisors are those for which Zhang's lower bound for the essential minimum is attained.

As we will see in §5, the condition for a toric metric of being quasi-canonical is very restrictive. The following observation is a direct consequence of Proposition 2.5 and of the inequality (2.7).

Proposition 2.8. *Let X be a proper variety over $\mathbb{K}$ of dimension n and $\overline{D}$ a semipositive metrized divisor on X with D big and semiample. Then there exists a generic net $(p_l)_{l\in I}$ of algebraic points of X with*

$$\lim_{l} \mathrm{h}_{\overline{D}}(p_l) = \frac{\mathrm{h}_{\overline{D}}(X)}{(n+1)\deg_D(X)} \tag{2.8}$$

if and only if $\overline{D}$ is quasi-canonical.

We discuss now the equidistribution of Galois orbits of points of small height.

Let X be a proper variety over $\mathbb{K}$ and $v \in \mathfrak{M}_{\mathbb{K}}$. We endow the space of probability measures on X_v^{an} with the weak-$*$ topology with respect to the space of continuous functions on X_v^{an}. In particular, a net of probability measures $(\mu_l)_{l\in I}$ converges to a probability measure μ if, for every continuous function $f\colon X_v^{\mathrm{an}} \to \mathbb{R}$,

$$\lim_{l} \int f \,\mathrm{d}\mu_l = \int f \,\mathrm{d}\mu.$$

Definition 2.9. Let $\overline{D}$ be a metrized $\mathbb{R}$-divisor on X. A probability measure μ on X_v^{an} is a *v-adic limit measure* for $\overline{D}$ if there exists a generic $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X such that the net of probability measures $(\mu_{p_l,v})_{l\in I}$ converges to μ. We say that $\overline{D}$ satisfies the *v-adic equidistribution property* if, for every generic $\overline{D}$-small net $(p_l)_{l\in I}$ as above, the net of measures $(\nu_{p_l,})_{l\in I}$ converges.

Clearly, when the v-adic equidistribution property holds, there exists a unique limit measure.

Remark 2.10. When $\mathbb{K}$ is a number field, the analytic space X_v^{an} is metrizable and so is the space of probability measures on it. Using this together with Remark 2.6 and standard arguments, one can reduce to sequences, instead of nets, when studying equidistribution properties over number fields.

In the literature there are many equidistribution theorems of Galois orbits of points of small height. All these equidistribution results deal with generic nets (or sequences when $\mathbb{K}$ is a number field) of algebraic points satisfying the equality (2.8). In view of Proposition 2.8, the existence of such a net implies that the metric is quasi-canonical. Moreover, the condition (2.8) for this net is equivalent of being $\overline{D}$-small. Thus we can reformulate a general equidistribution result in the following form.

Theorem 2.11. *Let $\mathbb{K}$ be a global field and X a projective variety over $\mathbb{K}$ of dimension n. Let $\overline{D}$ be a semipositive metrized divisor on X such that D is ample. If $\overline{D}$ is quasi-canonical then, for every place $v \in \mathfrak{M}_{\mathbb{K}}$,*

1. *$\overline{D}$ satisfies the v-adic equidistribution property;*
2. *the limit measure is the normalized Monge-Ampère measure*

$$\frac{1}{\deg_D(X)}\,\mathrm{c}_1(D, \|\cdot\|_v)^{\wedge n}.$$

This result is due to Yuan [Yua08, Theorem 3.1] in the number field case and, with more general hypotheses, to Gubler [Gub08, Theorem 1.1] in the function field case.

Written in this form, it is clear that this equidistribution theorem has a very restrictive hypothesis, that the metrized divisor $\overline{D}$ is quasi-canonical. But it also has a very strong thesis: not only the Galois orbits of points of small height converge to a measure, but this limit measure can be identified with the normalized Monge-Ampère measure of the metrized divisor.

The main objective of this paper is to start the study of what happens when the hypothesis of $\overline{D}$ being quasi-canonical is removed. We will work with toric

varieties and toric metrics because, in this case, the tools developed previously allow us to work very explicitly. In this setting, we will see that the first statement in Theorem 2.11 holds in much great generality, but, if the metric is not quasi-canonical, the limit measure does not need to agree with the normalized Monge-Ampère measure.

3. Auxiliary results on convex analysis

In this section we gather several definitions and results on convex analysis that we will use in our study of toric height functions. For a background in convex analysis, see for instance [BPS14a, §2].

Let $N_{\mathbb{R}} \simeq \mathbb{R}^n$ be a real vector space of dimension n and $M_{\mathbb{R}} = \operatorname{Hom}(N_{\mathbb{R}}, \mathbb{R}) = N_{\mathbb{R}}^{\vee}$ its dual. The pairing between $x \in M_{\mathbb{R}}$ and $u \in N_{\mathbb{R}}$ will be denoted either by $\langle x, u\rangle$ or $\langle u, x\rangle$.

Following [BPS14a, §2], a convex subset C is nonempty. The *relative interior* of C, denoted by $\operatorname{ri}(C)$, is the interior C relative to the minimal affine subspace containing it.

Let $C \subset M_{\mathbb{R}}$ be a convex subset and $g\colon C \to \mathbb{R}$ a concave function. The *sup-differential* of g at a point $x \in C$ is

$$\partial g(x) = \{u \in N_{\mathbb{R}} \mid \langle u, z - x\rangle \geq g(z) - g(x) \text{ for all } z \in C\}.$$

It is a convex subset of $N_{\mathbb{R}}$. The *stability set* of g is the subset of $N_{\mathbb{R}}$ defined by

$$\operatorname{stab}(g) = \{u \in N_{\mathbb{R}} \mid u - g \text{ is bounded below}\}.$$

The *Legendre-Fenchel dual* of g is the function $g^{\vee}\colon \operatorname{stab}(g) \to \mathbb{R}$ defined by

$$g^{\vee}(u) = \inf_{x\in C} \langle u, x\rangle - g(x). \tag{3.1}$$

Let $E \subset N_{\mathbb{R}}$ be a convex subset. A nonempty subset $F \subset E$ is a *face* of E if every closed segment $S \subset E$ whose relative interior has nonempty intersection with F, is contained in F.

Lemma 3.1. *Let $C \subset M_{\mathbb{R}}$ be a compact convex subset and $g_1, g_2\colon C \to \mathbb{R}$ two continuous concave functions. Denote by $C_{\max}$ the convex subset of C of the points where $g_1 + g_2$ attains its maximum value and choose $x \in C_{\max}$. For $i = 1, 2$, consider the concave function $\widehat{\phi}_i\colon N_{\mathbb{R}} \to \mathbb{R}$ defined by*

$$\widehat{\phi}_i(u) = g_i^{\vee}(u) - \langle x, u\rangle + g_i(x). \tag{3.2}$$

Then

1. *if $x' \in \operatorname{ri}(C_{\max})$, then $\partial g_i(x')$ is a face of $\partial g_i(x)$, $i = 1, 2$;*
2. *$\partial g_1(x) \cap (-\partial g_2(x))$ is nonempty and does not depend on the choice of $x \in C_{\max}$;*
3. *the minimal face of $\partial g_1(x)$ containing $\partial g_1(x) \cap (-\partial g_2(x))$ does not depend on the choice of $x \in C_{\max}$;*
4. *the function $\widehat{\phi}_i$ is nonpositive and vanishes precisely on $\partial g_i(x)$.*

Proof. The restriction to $C_{\max}$ of the sum $g_1 + g_2$ is constant, and so the restrictions to this set of g_1 and g_2 are affine and with opposite slopes. In other words, there is $u_0 \in N_{\mathbb{R}}$ such that, for all $x_1, x_2 \in C_{\max}$,

$$g_1(x_2) - g_1(x_1) = \langle u_0, x_2 - x_1\rangle \quad \text{and} \quad g_2(x_2) - g_2(x_1) = -\langle u_0, x_2 - x_1\rangle. \tag{3.3}$$

For the statement (1), let $i = 1, 2$ and $u \in \partial g_i(x')$. Since x' is in the relative interior of $C_{\max}$, there exists $\varepsilon > 0$ such that $x' - \varepsilon(x - x') \in C_{\max}$. By the definition of the sup-differential, for all $z \in C$,

$$\langle u, z - x'\rangle \geq g_i(z) - g_i(x'). \tag{3.4}$$

By (3.4) and (3.3),

$$\begin{aligned}-\varepsilon\langle u, x-x'\rangle &= \langle u, x'-\varepsilon(x-x')-x'\rangle \\ &\geq g_i(x'-\varepsilon(x-x'))-g_i(x') \\ &= \langle u_0, -\varepsilon(x-x')\rangle = -\varepsilon(g_i(x)-g_i(x')).\end{aligned}$$

Hence $\langle u, x-x'\rangle \leq g_i(x)-g_i(x')$. By (3.4) applied to $z=x$, we have also the reverse inequality. Thus $\langle u, x-x'\rangle = g_i(x)-g_i(x')$, and it follows from (3.4) that, for all $z \in C$,

$$\langle u, z-x\rangle \geq g_i(z)-g_i(x).$$

Hence $u \in \partial g_i(x)$ and so $\partial g_i(x') \subset \partial g_i(x)$. Then [BPS14a, Proposition 2.2.8] implies that $\partial g_i(x')$ is a face of $\partial g_i(x)$.

To prove the statement (2) note that, since g_1+g_2 attains its maximum value at x, by [BPS14a, Proposition 2.3.6(2)]

$$0 \in \partial(g_1+g_2)(x) = \partial g_1(x)+\partial g_2(x).$$

Hence $\partial g_1(x)\cap(-\partial g_2(x)) \neq \emptyset$, as stated. Now let u be a point in this intersection. Then

$$\langle u, z-x\rangle \geq g_1(z)-g_1(x) \quad \text{and} \quad \langle -u, z-x\rangle \geq g_2(z)-g_2(x). \tag{3.5}$$

Choose $x'' \in C_{\max}$. Subtracting, from the inequalities (3.5) applied to $z=x''$, the identities (3.3) applied to $x_1=x$ and $x_2=x''$, we deduce that

$$\langle u-u_0, x''-x\rangle = 0.$$

Using this together with (3.4)and (3.5), we obtain

$$\langle u, z-x''\rangle \geq g_1(z)-g_1(x'') \quad \text{and} \quad \langle -u, z-x''\rangle \geq g_2(z)-g_2(x'').$$

Hence $u \in \partial g_1(x'')\cap(-\partial g_2(x''))$, as stated.

For the next statement, consider the convex set $B=\partial g_1(x)\cap(-\partial g_2(x))$ which, thanks to (2), does not depend on the choice of $x \in C_{\max}$. Denote by F_x the minimal face of $\partial g_1(x)$ containing it. By (1), it is enough to consider the case when $x \in \mathrm{ri}(C_{\max})$. By the same statement, the set $\partial g_2(x)$ does not depend on the choice of $x \in \mathrm{ri}(C_{\max})$, proving (3).

The statement (4) follows readily from [BPS14a, Lemma 2.2.6]. □

Definition 3.2. Let $C \subset M_{\mathbb{R}}$ be a compact convex subset and $g_1, g_2 \colon C \to \mathbb{R}$ two continuous concave functions. Let $C_{\max}$ be the convex subset of C of the points where g_1+g_2 attains its maximum value. Given $x \in C_{\max}$, we define the convex subset of $N_{\mathbb{R}}$

$$B(g_1,g_2) = \partial g_1(x)\cap(-\partial g_2(x))$$

and the convex subset

$$F(g_1,g_2) \subset \partial g_1(x)$$

as the minimal face of $\partial g_1(x)$ that contains $B(g_1,g_2)$. By Lemma 3.1, these convex subsets do not depend on the choice of x.

We also define the convex subset $A_i(g_1,g_2)=\partial g_i(x) \subset N_{\mathbb{R}}$, $i=1,2$. By the same lemma, this convex subset does not depend on the choice of the point x.

Remark 3.3. In the setting of Lemma 3.1, it is not always true that $\partial g_i(x) = \partial g_i(x')$ for $x, x' \in C_{\max}$. An example of this situation is when g_i is the zero function on the interval $[0,1] \subset \mathbb{R}$ and $x=0, x'=1/2$. Hence, in Definition 3.2 we cannot define the sets $A_i(g_1,g_2)$ using an arbitrary point $x \in C_{\max}$.

Lemma 3.4. *Let $C \subset M_{\mathbb{R}}$ be a compact convex subset with nonempty interior and $g_1, g_2 \colon C \to \mathbb{R}$ two concave functions. Then $B(g_1,g_2)$ is bounded and $F(g_1,g_2)$ contains no lines.*

Proof. The convex set $B(g_1, g_2)$ is not bounded if and only if it contains a ray, that is, a subset of the form $\mathbb{R}_{\geq 0}u_1 + u_2$ with $u_i \in N_\mathbb{R}$, $i = 1, 2$, and $u_1 \neq 0$. Suppose that this is the case. This implies that, for $x \in C_{\max}$ and all $t \geq 0$,

$$tu_1 + u_2 \in \partial g_1(x) \quad \text{and} \quad -tu_1 - u_2 \in \partial g_2(x).$$

Hence, for all $z \in C$ and $t \geq 0$,

$$-\langle u_2, z - x\rangle + g_1(z) - g_1(x) \leq t\langle u_1, z - x\rangle \leq -\langle u_2, z - x\rangle - g_2(z) + g_2(x).$$

Letting $t \to \infty$, this implies $C \subset \{z \mid \langle u_1, z - x\rangle = 0\}$, contradicting the hypothesis that C has nonempty interior. Hence $B(g_1, g_2)$ is bounded.

Similarly, if $F(g_1, g_2)$ contains a line $\mathbb{R}u_1 + u_2$, then, for $x \in C_{\max}$ and $t \in \mathbb{R}$,

$$tu_1 + u_2 \in \partial g_1(x).$$

This also implies that C is contained in the affine hyperplane $\{z \mid \langle u_1, z - x\rangle = 0\}$ and contradicts the hypothesis that C has nonempty interior. Hence $F(g_1, g_2)$ contains no lines. □

Let $\mathcal{C}_\mathrm{b}(N_\mathbb{R})$ be the space of bounded continuous functions on $N_\mathbb{R}$, and let $\|\cdot\|$ be an auxiliary norm on $N_\mathbb{R}$ that we fix.

Definition 3.5. We denote by $\mathcal{P}$ the space of Borel probability measures on $N_\mathbb{R}$ endowed with the weak-$*$ topology with respect to $\mathcal{C}_\mathrm{b}(N_\mathbb{R})$. This is the smallest topology on $\mathcal{P}$ such that, for all $\varphi \in \mathcal{C}_\mathrm{b}(N_\mathbb{R})$, the function $\mu \mapsto \int \varphi \, \mathrm{d}\mu$ is continuous.

We denote by $\mathcal{E} \subset \mathcal{P}$ the topological subspace of probability measures with finite first moment, that is, the probability measures on $N_\mathbb{R}$ satisfying

$$\int \|u\| \, \mathrm{d}\mu(u) < \infty.$$

For $\mu \in \mathcal{E}$, the *expected value* is

$$\mathrm{E}[\mu] = \int u \, \mathrm{d}\mu(u) \in N_\mathbb{R}.$$

The weak-$*$ topology of $\mathcal{P}$ with respect to $\mathcal{C}_\mathrm{b}(N_\mathbb{R})$ is called the "topologie étroite" in [Bou69, §5]. By Proposition 5.4.10 in *loc. cit.*, the topological space $\mathcal{P}$ is complete, metrizable and separable. Latter we will consider other topologies on the underlying spaces of $\mathcal{P}$ and $\mathcal{E}$. When this is the case, we will state explicitly the used topology.

For $\mu \in \mathcal{P}$, its *support*, denoted by $\mathrm{supp}(\mu)$, is the set of all points in $N_\mathbb{R}$ such that all its neighborhoods have positive measure. The set of probability measures on $N_\mathbb{R}$ with finite support is contained in $\mathcal{E}$, and is dense therein.

For the rest of this section, we fix a compact convex subset $C \subset M_\mathbb{R}$ with nonempty interior and two continuous concave functions $g_1, g_2 \colon C \to \mathbb{R}$. The Legendre-Fenchel dual $g_i^\vee$ is a concave function on $N_\mathbb{R}$ with stability set C.

We introduce the function $\Phi \colon \mathcal{E} \to \mathbb{R}$ given, for $\mu \in \mathcal{E}$, by

$$\Phi(\mu) = \int g_1^\vee \, \mathrm{d}\mu + g_2^\vee(-\mathrm{E}[\mu]) + \max_{x \in C}(g_1(x) + g_2(x)). \tag{3.6}$$

This function will play a central role in the proof of the main results in the next section.

It follows easily from its definition that Φ is concave. In general, this function is not continuous, as the following example shows.

Example 3.6. Let $N_\mathbb{R} = \mathbb{R}$, so that $M_\mathbb{R} = \mathbb{R}$. Set $C = [0, 1]$ and $g_i = 0$, $i = 1, 2$. Then $g_i^\vee(u) = \min(0, u)$ for $u \in \mathbb{R}$. Consider the sequence of measures

$$\mu_l = \frac{l-1}{l}\delta_0 + \frac{1}{l}\delta_{-l}, \quad l \geq 1,$$

where δ_0 and δ_{-l} are the Dirac measures at the points 0 and $-l$, respectively. This sequence converges to δ_0. However, $\Phi(\mu_l) = -1$ for all l and $\Phi(\delta_0) = 0$.

Nevertheless, we have the following result.

Proposition 3.7. *The function Φ is upper semicontinuous.*

To prove this proposition, we need the following lemma.

Lemma 3.8. *Let $\phi\colon N_{\mathbb{R}} \to \mathbb{R}$ be a continuous function. If ϕ is bounded above (respectively below), then the map $\mathcal{P} \to \mathbb{R} \cup \{-\infty\}$ (respectively $\mathcal{P} \to \mathbb{R} \cup \{\infty\}$) given by*

$$\mu \longmapsto \int \phi \, \mathrm{d}\mu$$

is upper semicontinuous (respectively lower semicontinuous).

Proof. We prove the case of a function bounded above, the other case being analogous. Let $\mu \in \mathcal{P}$ and $\varepsilon > 0$ be given and, for $l \geq 1$, put

$$\phi_l(u) = \max(\phi(u), -l).$$

The sequence of functions $(\phi_l)_{l \geq 1}$ is monotone and converges pointwise to ϕ. So Lebesgue's monotone convergence theorem implies that there is $l_0 \geq 1$ such that

$$\int \phi_{l_0} \, \mathrm{d}\mu \leq \int \phi \, \mathrm{d}\mu + \varepsilon.$$

Let $(\mu_l)_{l \geq 1}$ be a sequence in $\mathcal{P}$ converging to μ. Since $\phi_{l_0} \in \mathcal{C}_{\mathrm{b}}(N_{\mathbb{R}})$, there exists $l_1 \geq 1$ such that, for $l \geq l_1$,

$$\int \phi \, \mathrm{d}\mu_l \leq \int \phi_{l_0} \, \mathrm{d}\mu_l \leq \int \phi_{l_0} \, \mathrm{d}\mu + \varepsilon \leq \int \phi \, \mathrm{d}\mu + 2\varepsilon.$$

Since ε is arbitrary, $\limsup_{l \to \infty} \int \phi \, \mathrm{d}\mu_l \leq \int \phi \, \mathrm{d}\mu$, proving the lemma. ☐

Proof of Proposition 3.7. Set $\phi_i = g_i^{\vee}$, $i = 1, 2$ for short. Fix $\mu_0 \in \mathcal{E}$ and set $u_0 = -\operatorname{E}[\mu_0] \in N_{\mathbb{R}}$. Choose $x \in \partial\phi_2(u_0) \subset M_{\mathbb{R}}$ so that, for all $u \in N_{\mathbb{R}}$,

$$\langle x, u - u_0 \rangle \geq \phi_2(u) - \phi_2(u_0).$$

Let $\mu \in \mathcal{E}$. It follows from this inequality that

$$\begin{aligned}
\Phi(\mu) - \Phi(\mu_0) &= \int \phi_1 \, \mathrm{d}\mu + \phi_2(-\operatorname{E}[\mu]) - \int \phi_1 \, \mathrm{d}\mu_0 - \phi_2(-\operatorname{E}[\mu_0]) \\
&\leq \int \phi_1 \, \mathrm{d}(\mu - \mu_0) - \langle \operatorname{E}[\mu] - \operatorname{E}[\mu_0], x \rangle \\
&= \int \phi_1 \, \mathrm{d}(\mu - \mu_0) - \int \langle u, x \rangle \, \mathrm{d}(\mu - \mu_0) \\
&= \int \phi \, \mathrm{d}(\mu - \mu_0)
\end{aligned}$$

with $\phi = \phi_1 - x$. Hence

$$\Phi(\mu) \leq \Phi(\mu_0) - \int \phi \, \mathrm{d}\mu_0 + \int \phi \, \mathrm{d}\mu. \tag{3.7}$$

Since x belongs to $\partial\phi_2(u_0)$ and $\partial\phi_2(u_0) \subset \operatorname{stab}(\phi_2) = \operatorname{stab}(\phi_1) = C$, the function ϕ is bounded above. By Lemma 3.8, the right-hand side of (3.7) is upper semicontinuous. The inequality (3.7) is an equality for $\mu = \mu_0$. Hence Φ is upper semicontinuous at μ_0, as stated. ☐

Proposition 3.9. *The function Φ is nonpositive, and vanishes for $\mu \in \mathcal{E}$ if and only if*

$$\mathrm{supp}(\mu) \subset F(g_1, g_2) \quad \textit{and} \quad \mathrm{E}[\mu] \in B(g_1, g_2), \tag{3.8}$$

with $B(g_1, g_2)$ and $F(g_1, g_2)$ as in Definition 3.2.

Proof. Let notation be as in Lemma 3.1 and for short put

$$A_i = A_i(g_1, g_2), \quad B = B(g_1, g_2), \quad F = F(g_1, g_2).$$

Fix a point $x \in \mathrm{ri}(C_{\max})$ and let $\widehat{\phi}_i$ be as in (3.2). For every $\mu \in \mathcal{E}$ we can write $\Phi(\mu)$ in terms of the functions $\widehat{\phi}_i$ as

$$\Phi(\mu) = \int \widehat{\phi}_1 \, \mathrm{d}\mu + \widehat{\phi}_2(-\mathrm{E}[\mu]). \tag{3.9}$$

By Lemma 3.1(4), the functions $\widehat{\phi}_i$ are nonpositive and vanish precisely on the sets A_i. It follows from (3.9) that Φ is nonpositive and vanishes for every $\mu \in \mathcal{E}$ satisfying (3.8).

Conversely, let $\mu \in \mathcal{E}$ such that $\Phi(\mu) = 0$. Since both $\widehat{\phi}_1$ and $\widehat{\phi}_2$ are nonpositive, the equality (3.9) also implies that

$$\int \widehat{\phi}_1 \, \mathrm{d}\mu = 0 \quad \text{and} \quad \widehat{\phi}_2(-\mathrm{E}[\mu]) = 0.$$

Therefore $\mathrm{supp}(\mu) \subset A_1$ and $-\mathrm{E}[\mu] \in A_2$. Since A_1 is convex, $\mathrm{E}[\mu] \in A_1$ and so

$$\mathrm{E}[\mu] \in A_1 \cap (-A_2) = B,$$

which gives the second condition in (3.8).

We next prove that the first condition in (3.8) is satisfied. Write $\theta = \mu(F)$, so that $0 \le \theta \le 1$ and $\mu(A_1 \setminus F) = 1 - \theta$.

If $\theta < 1$, we put

$$u_2 = \frac{1}{1-\theta} \int_{A_1 \setminus F} u \, \mathrm{d}\mu \in A_1 \setminus F.$$

If $\theta = 0$, then $\mathrm{E}[\mu] = u_2$ and so $\mathrm{E}[\mu] \in A_1 \setminus F$, contradicting the fact that $\mathrm{E}[\mu] \in B \subset F$. Suppose that $0 < \theta < 1$ and set

$$u_1 = \frac{1}{\theta} \int_F u \, \mathrm{d}\mu \in F.$$

Therefore

$$\mathrm{E}[\mu] = \theta u_1 + (1-\theta) u_2 \in \mathrm{ri}(\overline{u_1 u_2}),$$

the relative interior of the segment $\overline{u_1 u_2}$. Since $\mathrm{E}[\mu]$ is in B and hence in F, we have $\mathrm{ri}(\overline{u_1 u_2}) \cap F \neq \emptyset$. Moreover, the whole segment is contained in A_1, and F is a face of A_1. We deduce that this segment is contained in F. Therefore $u_2 \in F$, contradicting the fact that $u_2 \in A_1 \setminus F$. We conclude that $\theta = 1$ and so $\mathrm{supp}(\mu) \subset F$. This proves the first condition and completes the proof. □

The function Φ satisfies also the following property.

Lemma 3.10. *There are constants $c_1 \ge 0$ and $c_2 > 0$ such that, for all $\mu \in \mathcal{E}$,*

$$\Phi(\mu) \le c_1 - c_2 \int \|u\| \, \mathrm{d}\mu.$$

Proof. Let Ψ be the support function of C, which is the function on $N_{\mathbb{R}}$ given by

$$\Psi(u) = \min_{y \in C} \langle u, y \rangle.$$

Put $c_1 = 4 \max_{y \in C}(|g_1(y)|, |g_2(y)|)$. It follows from their definition that the functions $\phi_i = g_i^{\vee}$ verify, for $u \in N_{\mathbb{R}}$,

$$\max\left(\phi_1(u), \phi_2(u)\right) \le \Psi(u) + \frac{c_1}{4}. \tag{3.10}$$

Let x be a point in the interior of C. On $M_{\mathbb{R}}$, we consider the norm induced by the chosen norm $\|\cdot\|$ in $N_{\mathbb{R}}$. Since x is interior, we can find a constant $c_2 > 0$ such that $\mathcal{B}(x, c_2)$, the closed ball of center x and radius c_2, is contained in C. Then

$$\Psi(u) \leq \min_{y \in \mathcal{B}(x,c_2)} \langle u, y\rangle = \langle u, x\rangle - c_2\|u\|. \tag{3.11}$$

Since $x \in C = \mathrm{stab}(\Psi)$, we have $(\Psi - x)(u) \leq 0$. By (3.10) and (3.11),

$$\begin{aligned} \Phi(\mu) &= \int \phi_1(u)\,\mathrm{d}\mu + \phi_2(-\mathrm{E}[\mu]) + \max_{y\in C}(g_1(y) + g_2(y)) \\ &\leq c_1 + \int \Psi(u)\,\mathrm{d}\mu + \Psi(-\mathrm{E}[\mu]) \\ &= c_1 + \int (\Psi - x)(u)\,\mathrm{d}\mu + (\Psi - x)(-\mathrm{E}[\mu]) \\ &\leq c_1 - c_2 \int \|u\|\,\mathrm{d}\mu, \end{aligned}$$

as stated. □

Proposition 3.11. *Let $(\mu_l)_{l\in I}$ be a net of measures in $\mathcal{E}$ such that*

$$\lim_l \Phi(\mu_l) = 0.$$

Then $(\mu_l)_{l\in I}$ has at least one cluster point in $\mathcal{P}$, and every such cluster point μ lies in $\mathcal{E}$ and satisfies

$$\mathrm{supp}(\mu) \subset F(g_1, g_2) \quad \textit{and} \quad \mathrm{E}[\mu] \in B(g_1, g_2).$$

Proof. Replacing $(\mu_l)_{l\in I}$ by a subnet if necessary, we assume that $\Phi(\mu_l) \geq -1$ for all $l \in I$. Let c_1, c_2 be the constants of Lemma 3.10 and set $K = (c_1 + 1)/c_2 > 0$. This lemma implies that each μ_l is in the subset of $\mathcal{E}$ given by

$$\Big\{\mu \in \mathcal{E} \,\Big|\, \int \|u\|\,\mathrm{d}\mu(u) \leq K\Big\}.$$

This subset is compact thanks to Prokhorov's theorem [Bou69, Théorème 5.5.1], and it is metrizable because $\mathcal{P}$ is. Hence, the net $(\mu_l)_{l\in I}$ has at least one cluster point, and every such cluster point μ lies in $\mathcal{E}$, proving the first statement.

To prove the last statement, let $(\mu_k)_{k\in I'}$ be a subnet converging to μ. By Proposition 3.7, the function Φ is upper-semicontinuous and so

$$\Phi(\mu) \geq \limsup_k \Phi(\mu_k) = 0.$$

Hence $\Phi(\mu) = 0$, and the statement follows from Proposition 3.9. □

As we have seen in Example 3.6, the function Φ is not continuous. We now consider another topology on $\mathcal{E}$ with respect to which the function Φ is continuous.

Given $\mu, \mu' \in \mathcal{P}$, denote by $\Gamma(\mu, \mu')$ the set of probability measures on $N_{\mathbb{R}} \times N_{\mathbb{R}}$ with marginals μ and μ'. That is, a probability measure ν on $N_{\mathbb{R}} \times N_{\mathbb{R}}$ belongs to $\Gamma(\mu, \mu')$ if and only if

$$(p_1)_*\nu = \mu, \quad (p_2)_*\nu = \mu',$$

where p_i is the projection of $N_{\mathbb{R}} \times N_{\mathbb{R}}$ onto its i-th factor, and $(p_i)_*$ the direct image of measures.

Definition 3.12. The *Kantorovich–Rubinstein distance* (or *Wasserstein distance of order 1*) on $\mathcal{E}$ is defined, for $\mu, \mu' \in \mathcal{E}$, by

$$W(\mu, \mu') = \inf_{\nu\in\Gamma(\mu,\mu')} \int \|u - u'\|\,\mathrm{d}\nu(u, u').$$

The *Kantorovich–Rubinstein topology* (or KR-topology for short) of $\mathcal{E}$ is the topology induced by this distance.

For a Lipschitz continuous function $\psi\colon N_{\mathbb{R}} \to \mathbb{R}$, denote by $\mathrm{Lip}(\psi)$ its *Lipschitz constant*, given by

$$\mathrm{Lip}(\psi) = \sup_{u \neq u'} \frac{|\psi(u) - \psi(u')|}{\|u - u'\|}.$$

Lipschitz constants and the Kantorovich-Rubinstein distance are related by the *duality formula*: for $\mu, \mu' \in \mathcal{E}$ and a Lipschitz continuous function $\psi\colon N_{\mathbb{R}} \to \mathbb{R}$, we have

$$\left| \int \psi \,\mathrm{d}\mu - \int \psi \,\mathrm{d}\mu' \right| \leq \mathrm{Lip}(\psi)\, W(\mu, \mu'), \tag{3.12}$$

see for instance [Vil09, Remark 6.5].

Remark 3.13. By [Vil09, Theorem 6.9], the KR-topology agrees with the weak-$*$ topology on $\mathcal{E}$ with respect to the space of continuous functions $\varphi\colon N_{\mathbb{R}} \to \mathbb{R}$ such that

$$|\varphi(u)| \leq c(1 + \|u\|)$$

for a $c \in \mathbb{R}$ and all $u \in N_{\mathbb{R}}$. In particular, the KR-topology is stronger than the topology of $\mathcal{E}$ induced by that of $\mathcal{P}$ as in Definition 3.5.

Proposition 3.14. *The function Φ is continuous with respect to the KR-topology. In particular, if $(\mu_l)_{l \in I}$ is a net of measures in $\mathcal{E}$ that converges to a measure $\mu \in \mathcal{E}$ with respect to the KR-topology and*

$$\mathrm{supp}(\mu) \subset F(g_1, g_2) \quad \textit{and} \quad \mathrm{E}[\mu] \in B(g_1, g_2),$$

then $\lim_l \Phi(\mu_l) = 0$.

Proof. Let $(\mu_l)_{l \in I}$ be a net on $\mathcal{E}$ that converges to a measure $\mu \in \mathcal{E}$ with respect to the KR-topology. By Remark 3.13,

$$\lim_l \int g_1^{\vee} \,\mathrm{d}\mu_l = \int g_1^{\vee} \,\mathrm{d}\mu \quad \text{and} \quad \lim_l g_2^{\vee}(-E[\mu_l]) = g_2^{\vee}(-E[\mu]).$$

Therefore $\lim_l \Phi(\mu_l) = \Phi(\mu)$ and so Φ is continuous, proving the first statement. The second statement follows from the first one and Proposition 3.9. ☐

We also need the following easy result, that we include here for the lack of a suitable reference.

Lemma 3.15. *Let $E_i \subset N_{\mathbb{R}}$, $i = 1, \dots, r$, be convex subsets and $E = E_1 + \dots + E_r$ their Minkowski sum. For a point $u_0 \in E$, the following conditions are equivalent:*

1. *the point u_0 is a vertex of E;*
2. *the equation $u_0 = \sum_i z_i$ with $z_i \in E_i$ has a unique solution and, for $i = 1, \dots, r$, the point z_i in this solution is a vertex of E_i.*

Proof. First assume that u_0 is a vertex of E. Suppose that the equation $u_0 = \sum_i z_i$, $z_i \in E_i$, has two different solutions, namely $u_0 = \sum_i z_i'$ and $u_0 = \sum_i z_i''$ with $z_{i_0}' \neq z_{i_0}''$ for some $i_0 \in \{1, \dots, r\}$. Then both points

$$u_1 = \sum_{i \neq i_0} z_i' + z_{i_0}'' \quad \text{and} \quad u_2 = \sum_{i \neq i_0} z_i'' + z_{i_0}'$$

belong to E, they are different and satisfy $u_0 = \frac{1}{2}(u_1 + u_2)$, contradicting the fact that u_0 is a vertex of E. Hence the equation $u_0 = \sum_i z_i$ has a unique solution with $z_i \in E_i$.

Now suppose that z_{i_0} is not a vertex of E_{i_0} for some $i_0 \in \{1, \dots, r\}$. Then we can write $z_{i_0} = \frac{1}{2}(z'_{i_0} + z''_{i_0})$ with $z'_{i_0} \neq z''_{i_0}$ both in E_{i_0}. Hence the points

$$u_1 = \sum_{i \neq i_0} z_i + z'_{i_0} \quad \text{and} \quad u_2 = \sum_{i \neq i_0} z_i + z''_{i_0}$$

are different, belong to E and $u_0 = \frac{1}{2}(u_1 + u_2)$, contradicting the assumption that u_0 is a vertex of E. Thus we have proved that (1) implies (2).

Assume now that the statement (2) is true but u_0 is not a vertex of E. Then there are two different points $u_1, u_2 \in E$ with $u_0 = \frac{1}{2}(u_1 + u_2)$. Since E is the Minkowski sum of the sets E_i, we can write

$$u_0 = \sum_i z_i, \quad u_1 = \sum_i z'_i \quad \text{and} \quad u_2 = \sum_i z''_i.$$

The equation $u_0 = \sum_i z_i$ has a unique solution and so $z_i = \frac{1}{2}(z'_i + z''_i)$ for all i. Since z_i is a vertex of E_i, this implies $z'_i = z''_i$. Therefore $u_1 = u_2$, contradicting the assumptions and thus proving that (2) implies (1). □

4. Modulus distribution

In this section, we study the asymptotic modulus distribution of the Galois orbits of nets of points of small height in toric varieties. Our approach is based on the techniques developed in the series of papers [BPS14a, BMPS12, BPS14b]. These techniques are well-suited for the study of toric metrics and their associated height functions. In the sequel, we recall the basic constructions and results.

Let $\mathbb{K}$ be a global field and $\mathbb{T} \simeq \mathbb{G}_{\mathrm{m},\mathbb{K}}^n$ a split torus of dimension n over $\mathbb{K}$. Let

$$N = \mathrm{Hom}(\mathbb{G}_{m,\mathbb{K}}, \mathbb{T}) \quad \text{and} \quad M = \mathrm{Hom}(\mathbb{T}, \mathbb{G}_{m,\mathbb{K}}) = N^\vee$$

be the lattices of cocharacters and of characters of $\mathbb{T}$, respectively, and write $N_\mathbb{R} = N \otimes \mathbb{R}$ and $M_\mathbb{R} = M \otimes \mathbb{R}$. We also fix an auxiliary norm $\|\cdot\|$ on $N_\mathbb{R}$.

Let $v \in \mathfrak{M}_\mathbb{K}$. We denote by $\mathbb{T}_v^{\mathrm{an}}$ the v-adic analytification of $\mathbb{T}$ and by $\mathbb{S}_v$ its compact subtorus. In the Archimedean case, $\mathbb{S}_v$ is isomorphic to $(S^1)^n$ whereas, in the non-Archimedean case, it is a compact analytic group, see [BPS14a, § 4.2] for a description. Moreover, there is a map $\mathrm{val}_v \colon \mathbb{T}_v^{\mathrm{an}} \to N_\mathbb{R}$, defined, in a given splitting, by

$$\mathrm{val}_v(x_1, \dots, x_n) = (-\log|x_1|_v, \dots, -\log|x_n|_v). \tag{4.1}$$

This map does not depend on the choice of the splitting, and the compact torus $\mathbb{S}_v$ coincides with its fiber over the point $0 \in N_\mathbb{R}$.

Let X be a proper toric variety over $\mathbb{K}$ with torus $\mathbb{T}$, described by a complete fan Σ on $N_\mathbb{R}$. To each cone $\sigma \in \Sigma$ corresponds an affine toric variety X_σ, which is an open subset of X, and an orbit $O(\sigma)$ of the action of $\mathbb{T}$ on X. The affine toric variety corresponding to the cone $\sigma = \{0\}$ is the *principal open subset* X_0. It coincides with the orbit $O(0)$ and is canonically isomorphic to the torus $\mathbb{T}$.

An $\mathbb{R}$-divisor D on X is *toric* if it is invariant under the action of $\mathbb{T}$. Such an $\mathbb{R}$-divisor defines a *virtual support function* on Σ, that is a function

$$\Psi_D \colon N_\mathbb{R} \longrightarrow \mathbb{R}$$

whose restriction to each cone of the fan Σ is linear. We also associate to D the subset of $M_\mathbb{R}$ given by

$$\Delta_D = \mathrm{stab}(\Psi_D) = \{x \in M_\mathbb{R} \mid x \geq \Psi_D\}.$$

If D is pseudo-effective, then Δ_D is a polytope and, otherwise, it is the empty set. Properties of the $\mathbb{R}$-divisor D can be read off from its associated virtual support function and polytope. In particular, D is nef if and only if Ψ_D is concave, and D is big if and only if Δ_D has nonempty interior.

A quasi-algebraic metrized divisor $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}_{\mathbb{K}}})$ on X is *toric* if and only if the v-adic metric $\|\cdot\|_v$ is invariant with respect to the action of $\mathbb{S}_v$, for all v. Such a toric metrized $\mathbb{R}$-divisor on X defines a family of continuous functions $\psi_{\overline{D},v}\colon N_{\mathbb{R}} \to \mathbb{R}$ indexed by the places of $\mathbb{K}$. For each $v \in \mathfrak{M}_{\mathbb{K}}$, this function is given, for $p \in \mathbb{T}_v^{\mathrm{an}}$, by

$$\psi_{\overline{D},v}(\mathrm{val}_v(p)) = \log\|s_D(p)\|_v, \tag{4.2}$$

where s_D is the canonical rational $\mathbb{R}$-section of D as in [BMPS12, § 3]. This adelic family of functions satisfies that $|\psi_{\overline{D},v} - \Psi_D|$ is bounded for all v, and that $\psi_{\overline{D},v} = \Psi_D$ for all v except for a finite number. In particular, the stability set of each $\psi_{\overline{D},v}$ coincides with Δ_D.

For each $v \in \mathfrak{M}_{\mathbb{K}}$, we also consider the *$v$-adic roof function* $\vartheta_{\overline{D},v}\colon \Delta_D \to \mathbb{R}$, that is given by

$$\vartheta_{\overline{D},v}(x) = \psi_{\overline{D},v}^{\vee}(x) = \inf_{u\in N_{\mathbb{R}}} (\langle u, x\rangle - \psi_{\overline{D},v}(u)).$$

This is an adelic family of continuous concave functions on Δ_D which are zero except for a finite number of places. The *global roof function* $\vartheta_{\overline{D}}\colon \Delta_D \to \mathbb{R}$ is the weighted sum

$$\vartheta_{\overline{D}} = \sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \vartheta_{\overline{D},v}.$$

The essential minimum of X with respect to $\overline{D}$ defined in (2.6) can be computed as the maximum of its roof function [BPS14b, Theorem 1.1], that is

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) = \max_{x\in\Delta_D} \vartheta_{\overline{D}}(x). \tag{4.3}$$

Example 4.1. Let X be a proper toric variety over $\mathbb{K}$ and D a toric $\mathbb{R}$-divisor on X. The *canonical metric* on D is the metric characterized by the fact that, for each $v \in \mathfrak{M}_{\mathbb{K}}$ and $p \in \mathbb{T}_v^{\mathrm{an}}$,

$$\log\|s_D(p)\|_{\mathrm{can},v} = \Psi_D(\mathrm{val}_v(p)),$$

see [BPS14a, Proposition-Definition 4.3.15]. We denote this toric metrized $\mathbb{R}$-divisor by $\overline{D}^{\mathrm{can}}$. For all $v \in \mathfrak{M}_{\mathbb{K}}$,

$$\psi_{\overline{D}^{\mathrm{can}},v} = \Psi_D \quad \text{and} \quad \vartheta_{\overline{D}^{\mathrm{can}},v} = 0.$$

In particular, $\vartheta_{\overline{D}^{\mathrm{can}}} = 0$ and $\upmu^{\mathrm{ess}}_{\overline{D}^{\mathrm{can}}}(X) = 0$.

Given a semipositive toric metrized $\mathbb{R}$-divisor $\overline{D}$ over D, its associated metric functions are concave. Conversely, every adelic family of concave continuous functions $\psi_v\colon N_{\mathbb{R}} \to \mathbb{R}$, $v \in \mathfrak{M}_K$, with $|\psi_v - \Psi_D|$ bounded for all v and such that $\psi_{\overline{D},v} = \Psi_D$ for all v except for a finite number, corresponds to a semipositive toric metrized $\mathbb{R}$-divisor over D [BMPS12, Proposition 4.19(1)]. For instance, a canonical toric metrized $\mathbb{R}$-divisor $\overline{D}^{\mathrm{can}}$ is semipositive if and only if Ψ_D is concave, which is equivalent to the condition that D is nef.

For the rest of this section, we suppose that X is a proper toric variety over the global field $\mathbb{K}$ with torus $\mathbb{T}$, and that $\overline{D}$ is a semipositive toric metrized $\mathbb{R}$-divisor with D big.

We also fix the notation below. Recall from §3 that $\mathcal{P}$ denotes the space of probability measures on $N_{\mathbb{R}}$ endowed with the weak-$*$ topology with respect to the space $\mathcal{C}_{\mathrm{b}}(N_{\mathbb{R}})$, and that $\mathcal{E}$ denotes the subspace of probability measures with finite first moment.

Notation 4.2. Let $v \in \mathfrak{M}_{\mathbb{K}}$. We denote by $g_{i,v}$, $i = 1, 2$, the concave functions on Δ_D given by

$$g_{1,v} = \vartheta_{\overline{D},v} \quad \text{and} \quad g_{2,v} = \sum_{w\in\mathfrak{M}_{\mathbb{K}}\setminus\{v\}} \frac{n_w}{n_v}\vartheta_{\overline{D},w}.$$

Thus $\vartheta_{\overline{D}} = n_v(g_{1,v} + g_{2,v})$. We consider the convex subsets of $N_{\mathbb{R}}$ given by Definition 3.2

$$B_v = B(g_{1,v}, g_{2,v}),\ F_v = F(g_{1,v}, g_{2,v}) \quad \text{and} \quad A_v = A_1(g_{1,v}, g_{2,v}). \tag{4.4}$$

Recall that F_v is the minimal face of A_v containing B_v. We also denote by Φ_v the function on $\mathcal{E}$ obtained by applying Definition (3.6) to the set $C = \Delta_D$ and the functions $g_{i,v}$, $i = 1, 2$.

Given $v \in \mathfrak{M}_{\mathbb{K}}$ and a point $p \in X(\overline{\mathbb{K}})$, we consider the discrete probability measure on $N_{\mathbb{R}}$ defined by

$$\nu_{p,v} = (\mathrm{val}_v)_* \mu_{p,v},$$

where $\mu_{p,v}$ is the uniform discrete probability measure on X_v^{an} supported on the v-adic Galois orbit of p as in (2.4). This probability measure on $N_{\mathbb{R}}$ gives the modulus distribution of the v-adic Galois orbit of the point p. The next result characterizes the limit behavior of this modulus distribution for nets of points of small height.

Theorem 4.3. *Let notation and hypothesis be as above. For each $v \in \mathfrak{M}_{\mathbb{K}}$ and every $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points in the principal open subset X_0, the net $(\nu_{p_l,v})_{l\in I}$ of measures in $\mathcal{P}$ has at least one cluster point. Every such cluster point ν_v lies in $\mathcal{E}$ and satisfies*

$$\mathrm{supp}(\nu_v) \subset F_v \quad \textit{and} \quad \mathrm{E}[\nu_v] \in B_v. \tag{4.5}$$

The proof of Theorem 4.3 is given below, after a definition and an auxiliary result.

Definition 4.4. A *centered adelic measure* $\boldsymbol{\nu}$ on $N_{\mathbb{R}}$ is a collection of measures $\nu_v \in \mathcal{E}$, $v \in \mathfrak{M}_{\mathbb{K}}$, such that $\nu_v = \delta_0$, the Dirac measure at the point $0 \in N_{\mathbb{R}}$, for all but a finite number of places v, and such that

$$\sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \,\mathrm{E}[\nu_v] = 0. \tag{4.6}$$

We denote by $\mathcal{H}_{\mathbb{K}}$ the set of all centered adelic measures on $N_{\mathbb{R}}$.

We introduce the function $\eta_{\overline{D}} \colon \mathcal{H}_{\mathbb{K}} \to \mathbb{R}$ defined by

$$\eta_{\overline{D}}(\boldsymbol{\nu}) = -\sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \int \psi_{\overline{D},v}\, \mathrm{d}\nu_v. \tag{4.7}$$

This function extends the notion of height of points to the space $\mathcal{H}_{\mathbb{K}}$. Indeed, for $p \in X_0(\overline{\mathbb{K}})$, the collection

$$\boldsymbol{\nu}_p = (\nu_{p,v})_{v\in\mathfrak{M}_{\mathbb{K}}} \tag{4.8}$$

is a centered adelic measure on $N_{\mathbb{R}}$, because of the product formula in Proposition 2.1(2). Moreover, the canonical $\mathbb{R}$-section s_D does not vanish at p and, by Proposition 2.3 and (4.2),

$$\begin{aligned} \mathrm{h}_{\overline{D}}(p) &= -\sum_v \frac{n_v}{\#\,\mathrm{Gal}(p)_v} \sum_{q\in\mathrm{Gal}(p)_v} \psi_{\overline{D},v}(\mathrm{val}_v(q)) \\ &= -\sum_v n_v \int \psi_{\overline{D},v}\, \mathrm{d}\nu_{p,v} = \eta_{\overline{D}}(\boldsymbol{\nu}_p). \end{aligned} \tag{4.9}$$

Lemma 4.5. *For every centered adelic measure $\boldsymbol{\nu} = (\nu_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$,*

$$\max_{v\in\mathfrak{M}_{\mathbb{K}}} -n_v\Phi_v(\nu_v) \le \eta_{\overline{D}}(\boldsymbol{\nu}) - \mu^{\mathrm{ess}}_{\overline{D}}(X) \le \sum_{v\in\mathfrak{M}_{\mathbb{K}}} -n_v\Phi_v(\nu_v). \tag{4.10}$$

In particular, for $p \in X_0(\overline{\mathbb{K}})$,

$$\max_{v\in\mathfrak{M}_{\mathbb{K}}} -n_v\Phi_v(\nu_{p,v}) \le \mathrm{h}_{\overline{D}}(p) - \upmu^{\mathrm{ess}}_{\overline{D}}(X) \le \sum_{v\in\mathfrak{M}_{\mathbb{K}}} -n_v\Phi_v(\nu_{p,v}). \tag{4.11}$$

Proof. Let $\Delta_{D,\max}$ be the set of points maximizing the roof function $\vartheta_{\overline{D}}$ and choose $x \in \Delta_{D,\max}$. For each $v \in \mathfrak{M}_{\mathbb{K}}$, let $\widehat{\phi}_{i,v}\colon N_{\mathbb{R}} \to \mathbb{R}$, $i = 1, 2$, be the function defined by

$$\widehat{\phi}_{i,v}(u) = g^{\vee}_{i,v}(u) - \langle x, u\rangle + g_{i,v}(x),$$

where $g_{i,v}$ denotes the concave function on Δ_D in Notation 4.2 and $g^{\vee}_{i,v}$ its Legendre dual as in (3.1).

Note that $\psi_{\overline{D},v} = g^{\vee}_{1,v}$. Using (4.6) and (4.3), we deduce that

$$-\sum_v n_v \int \psi_{\overline{D},v}\,\mathrm{d}\nu_v = \vartheta_{\overline{D}}(x) - \sum_v n_v \int \widehat{\phi}_{1,v}\,\mathrm{d}\nu_v = \upmu^{\mathrm{ess}}_{\overline{D}}(X) - \sum_v n_v \int \widehat{\phi}_{1,v}\,\mathrm{d}\nu_v.$$

Thus

$$\eta_{\overline{D}}(\boldsymbol{\nu}) - \upmu^{\mathrm{ess}}_{\overline{D}}(X) = -\sum_v n_v \int \widehat{\phi}_{1,v}\,\mathrm{d}\nu_v. \tag{4.12}$$

For each $v \in \mathfrak{M}_{\mathbb{K}}$, we get from the definition of Φ_v that

$$\Phi_v(\nu_v) = \int \widehat{\phi}_{1,v}\,\mathrm{d}\nu_v + \widehat{\phi}_{2,v}(-\mathrm{E}[\nu_v]).$$

By Lemma 3.1(4), the functions $\widehat{\phi}_{i,v}$ are nonpositive and so

$$\Phi_v(\nu_v) \le \int \widehat{\phi}_{1,v}\,\mathrm{d}\nu_v. \tag{4.13}$$

The second inequality in (4.10) then follows from (4.12) and (4.13).

To prove the first inequality in (4.10), fix $v \in \mathfrak{M}_{\mathbb{K}}$. By [BPS14a, Propositions 2.3.1(1) and 2.3.3(3)],

$$\widehat{\phi}_{2,v} = \boxplus_{w\neq v}\Big(\widehat{\phi}_{1,w}\frac{n_w}{n_v}\Big), \tag{4.14}$$

where w runs over the places of $\mathbb{K}$ different from v, the symbol $\boxplus$ denotes the sup-convolution and, for a concave function ψ and a nonzero constant λ, the expression $\psi\lambda$ denotes the right multiplication as in [BPS14a, §2.3].

By the equality (4.14), the definitions of the sup-convolution and the right multiplication, and condition (4.6), we deduce

$$\widehat{\phi}_{2,v}(-\mathrm{E}[\nu_v]) \ge \sum_{w\neq v} \frac{n_w}{n_v}\widehat{\phi}_{1,w}(\mathrm{E}[\nu_w]). \tag{4.15}$$

By the concavity of $\widehat{\phi}_{1,w}$, we have $\int \widehat{\phi}_{1,w}\,\mathrm{d}\nu_w \le \widehat{\phi}_{1,w}(\mathrm{E}(\nu_w))$ for all $w \in \mathfrak{M}_{\mathbb{K}}$. Therefore, by (4.12) and (4.15),

$$\begin{aligned}\eta_{\overline{D}}(\boldsymbol{\nu}) - \upmu^{\mathrm{ess}}_{\overline{D}}(X) &\ge -n_v\bigg(\int \widehat{\phi}_{1,v}\,\mathrm{d}\nu_v + \sum_{w\neq v}\frac{n_w}{n_v}\widehat{\phi}_{1,w}(\mathrm{E}(\nu_w))\bigg)\\ &\ge -n_v\bigg(\int \widehat{\phi}_{1,v}\,\mathrm{d}\nu_v + \widehat{\phi}_{2,v}(-\mathrm{E}[\nu_v])\bigg) = -n_v\Phi_v(\nu_v),\end{aligned}$$

which proves the first inequality and completes the proof of (4.10). The inequalities in (4.11) follow directly from (4.10) and (4.9). □

Proof of Theorem 4.3. Let $v \in \mathfrak{M}_{\mathbb{K}}$ and $\Phi_v\colon \mathcal{E} \to \mathbb{R}$ the function defined by (3.6) with $g_{1,v}$ and $g_{2,v}$ as in Notation 4.2. Since the net of points $(p_l)_{l\in I}$ is $\overline{D}$-small,

$$\lim_l \mathrm{h}_{\overline{D}}(p_l) = \upmu^{\mathrm{ess}}_{\overline{D}}(X).$$

From Lemma 4.5, we deduce that

$$\lim_{l} \Phi_v(\nu_{p_l,v}) = 0.$$

The theorem is then a direct consequence of Proposition 3.11. □

To state a partial converse of Theorem 4.3, we need a further definition.

Definition 4.6. The *adelic Kantorovich–Rubinstein distance* $W_{\mathbb{K}}$ on $\mathcal{H}_{\mathbb{K}}$ is defined, for $\boldsymbol{\nu} = (\nu_v)_v, \boldsymbol{\nu}' = (\nu'_v)_v \in \mathcal{H}_{\mathbb{K}}$, by

$$W_{\mathbb{K}}(\boldsymbol{\nu}, \boldsymbol{\nu}') = \sum_v n_v W(\nu_v, \nu'_v),$$

where W denotes the Kantorovich–Rubinstein distance in $N_{\mathbb{R}}$ as in Definition 3.12. By the definition of $\mathcal{H}_{\mathbb{K}}$, there is only a finite number of nonzero terms in this sum.

The topology on $\mathcal{H}_{\mathbb{K}}$ induced by this distance is called the *adelic KR-topology.*

Theorem 4.7. *With notation and hypothesis as before, let $\boldsymbol{\nu} = (\nu_v)_{v\in\mathfrak{M}_{\mathbb{K}}} \in \mathcal{H}_{\mathbb{K}}$ be a centered adelic measure such that*

$$\operatorname{supp}(\nu_v) \subset F_v \quad \text{and} \quad \mathrm{E}[\nu_v] \in B_v$$

for all v. Then there is a generic $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X_0 such that the net of measures $(\boldsymbol{\nu}_{p_l})_{l\in I}$ converges to $\boldsymbol{\nu}$ with respect to the adelic Kantorovich-Rubinstein distance.

The proof of Theorem 4.7 is given below, after some preliminary results. The first result gives the main properties of the function $\eta_{\overline{D}}$.

Lemma 4.8. *The function $\eta_{\overline{D}}$ is Lipschitz continuous with respect to $W_{\mathbb{K}}$. Moreover, for all $\boldsymbol{\nu} = (\nu_v)_{v\in\mathfrak{M}_{\mathbb{K}}} \in \mathcal{H}_{\mathbb{K}}$,*

$$\eta_{\overline{D}}(\boldsymbol{\nu}) \geq \mu^{\mathrm{ess}}_{\overline{D}}(X), \tag{4.16}$$

with equality if and only if $\operatorname{supp}(\nu_v) \subset F_v$ *and* $\mathrm{E}[\nu_v] \in B_v$ *for all v.*

Proof. Let $S \subset \mathfrak{M}_{\mathbb{K}}$ be a finite subset such that $\psi_{\overline{D},v} = \Psi_D$ for all $v \notin S$. For $\boldsymbol{\nu} = (\nu_v)_v, \boldsymbol{\nu}' = (\nu'_v)_v \in \mathcal{H}_{\mathbb{K}}$,

$$\begin{aligned}|\eta_{\overline{D}}(\boldsymbol{\nu}) - \eta_{\overline{D}}(\boldsymbol{\nu}')| &\leq \sum_v n_v \left| \int \psi_{\overline{D},v}\, \mathrm{d}\nu_v - \int \psi_{\overline{D},v}\, \mathrm{d}\nu'_v \right| \\ &\leq \sum_v \operatorname{Lip}(\psi_{\overline{D},v}) n_v W(\nu_v, \nu'_v) \leq \Big(\max_{x\in\Delta_D} \|x\| \Big) W_{\mathbb{K}}(\boldsymbol{\nu}, \boldsymbol{\nu}').\end{aligned}$$

where the second inequality is given by the duality formula (3.12) and the last by the observation that $\operatorname{Lip}(\psi_{\overline{D},v}) = \max_{x\in\Delta_D} \|x\|$ for all v. This proves that $\eta_{\overline{D}}$ is Lipschitz continuous with respect to $W_{\mathbb{K}}$.

As already remarked, the functions Φ_v are nonpositive. By Lemma 4.5, this implies the inequality (4.16). From the same result, it follows that the equality holds if and only if $\Phi_v(\nu_v) = 0$ for all v. By Proposition 3.9, this holds if and only if $\operatorname{supp}(\nu_v) \subset F_v$ and $\mathrm{E}[\nu_v] \in B_v$, completing the proof of the lemma. □

From this lemma, we deduce as a direct consequence the next characterization of algebraic points in toric varieties realizing the essential minimum.

Corollary 4.9. *Let p be an algebraic point of X_0. Then $\mathrm{h}_{\overline{D}}(p) = \mu^{\mathrm{ess}}_{\overline{D}}(X)$ if and only if* $\operatorname{supp}(\nu_{p,v}) \subset F_v$ *and* $\mathrm{E}[\nu_{p,v}] \in B_v$ *for all $v \in \mathfrak{M}_{\mathbb{K}}$.*

Let $H_{\mathbb{K}} \subset \bigoplus_{v\in\mathfrak{M}_{\mathbb{K}}} N_{\mathbb{R}}$ be the subspace defined by the equation $\sum_v n_v u_v = 0$. By sending the point $(u_v)_v \in H_{\mathbb{K}}$ to the adelic centered measure $(\delta_{u_v})_v \in \mathcal{H}_{\mathbb{K}}$, we identify $H_{\mathbb{K}}$ with a subspace of $\mathcal{H}_{\mathbb{K}}$.

Corollary 4.10. *The minimum of the function $\eta_{\overline{D}}$ is equal to $\mu^{\mathrm{ess}}_{\overline{D}}(X)$ and it is attained at a point of the subspace $H_{\mathbb{K}} \subset \mathcal{H}_{\mathbb{K}}$.*

Proof. We denote by $\Delta_{D,\max} \subset \Delta_D$ the set of points where $\vartheta_{\overline{D}}$ attains its maximum. Let $x \in \Delta_{D,\max}$. Since $\sum_v n_v \partial\vartheta_{\overline{D},v}(x) = \partial\vartheta_{\overline{D}}(x)$ and $0 \in \partial\vartheta_{\overline{D}}(x)$ we can find a point $\boldsymbol{u} = (u_v)_v \in H_{\mathbb{K}}$ such that $u_v \in \partial\vartheta_{\overline{D},v}(x)$. From the definition of B_v, it follows that $u_v \in B_v$ for every v. Thus, by Lemma 4.8,

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \eta_{\overline{D}}(\boldsymbol{u}) = \min_{\boldsymbol{\nu} \in \mathcal{H}_{\mathbb{K}}} \eta_{\overline{D}}(\boldsymbol{\nu}),$$

as stated. □

We next show that the measures coming from algebraic points are dense in $\mathcal{H}_{\mathbb{K}}$.

Proposition 4.11. *For every $\boldsymbol{\nu} \in \mathcal{H}_{\mathbb{K}}$ there is a generic net $(p_l)_{l \in I}$ of algebraic points of X_0 such that the net of associated measures $(\boldsymbol{\nu}_{p_l})_{l \in I}$ as in (4.8), converges to $\boldsymbol{\nu}$ with respect to the adelic KR-topology.*

Proof. Put $\boldsymbol{\nu} = (\nu_v)_v$ and let $\varepsilon > 0$ be given. Let S be a finite nonempty subset of $\mathfrak{M}_{\mathbb{K}}$ such that $\nu_v = \delta_0$ for all $v \notin S$. By [Vil09, Theorem 6.18], we can approach, with respect to the KR-distance, each ν_v, $v \in S$, by a probability measure with finite support. Therefore, we can find $d \geq 1$ big enough and, for each $v \in S$, a sequence $(u_{v,1}, \dots, u_{v,d})$ of points of $N_{\mathbb{R}}$, such that the probability measure $\nu'_v = \frac{1}{d}\sum_{i=1}^{d} \delta_{u_{v,i}}$ verifies

$$W(\nu_v, \nu'_v) < \frac{\varepsilon}{2\sum_{v \in S} n_v} \quad \text{and} \quad \mathrm{E}[\nu'_v] = \mathrm{E}[\nu_v].$$

Set also $\nu'_v = \delta_0$ for $v \notin S$. Then $\boldsymbol{\nu}' = (\nu'_v)_v \in \mathcal{H}_{\mathbb{K}}$ and $W_{\mathbb{K}}(\boldsymbol{\nu}, \boldsymbol{\nu}') < \frac{\varepsilon}{2}$.

Let $\mathbb{F}/\mathbb{K}$ be a finite extension of degree d such that all places in S split completely, as given by [BPS14b, Lemma 2.2]. For each $v \in S$ and $w \in \mathfrak{M}_{\mathbb{F}}$ such that $w \mid v$, we have $n_w = n_v/d$. We enumerate the places above a given place $v \in S$ as $w(v,j)$, $j = 1, \dots, d$.

Let $H_{\mathbb{F}} \subset \bigoplus_{w \in \mathfrak{M}_{\mathbb{F}}} N_{\mathbb{R}}$ be the subspace defined by the equation $\sum_w n_w u_w = 0$. For each $v \in \mathfrak{M}_{\mathbb{K}}$ consider the element $\boldsymbol{u} \in H_{\mathbb{F}}$ given, for $w \in \mathfrak{M}_{\mathbb{F}}$, by

$$u_w = \begin{cases} u_{v,j} & \text{for } v \in S \text{ and } w = w(v,j) \text{ with } 1 \leq j \leq d, \\ 0 & \text{for } v \notin S \text{ and } w \mid v. \end{cases}$$

Consider the map $\mathrm{val}_{\mathbb{F}} \colon \mathbb{T}(\mathbb{F}) \to \bigoplus_{w \in \mathfrak{M}_{\mathbb{F}}} N_{\mathbb{R}}$ defined by $\mathrm{val}_{\mathbb{F}} = (\mathrm{val}_w)_{w \in \mathfrak{M}_{\mathbb{F}}}$. This is a group homomorphism and so it can be extended to a map

$$\mathrm{val}_{\mathbb{F}} \colon \mathbb{T}(\mathbb{F}) \otimes \mathbb{Q} \longrightarrow \bigoplus_{w \in \mathfrak{M}_{\mathbb{F}}} N_{\mathbb{R}}.$$

By the product formula, the image of this map lies in the hyperplane $H_{\mathbb{F}}$ and, by [BPS14b, Lemma 2.3], it is dense with respect to the L^1-topology on $H_{\mathbb{F}}$. For $\alpha \in \mathbb{T}(\mathbb{F})$ and $r \in \mathbb{Q}$, we have

$$\begin{aligned} \|\boldsymbol{u} - \mathrm{val}_{\mathbb{F}}(\alpha^r)\|_{L^1} &= \sum_{v \in S} \frac{n_v}{d} \sum_{j=1}^{d} \|u_{v,j} - \mathrm{val}_{w(v,j)}(\alpha^r)\| + \sum_{v \notin S} \|\mathrm{val}_v(\alpha^r)\| \\ &= \sum_v n_v \int \|u - u'\| \,\mathrm{d}\lambda_v(u,u') \qquad (4.17) \end{aligned}$$

for the probability measure λ_v on $N_{\mathbb{R}} \times N_{\mathbb{R}}$ given by

$$\lambda_v = \begin{cases} \dfrac{1}{d}\displaystyle\sum_{j=1}^{d} \delta_{(u_{v,j}, \mathrm{val}_{w(v,j)}(\alpha^r))} & \text{if } v \in S, \\ \delta_{(0, \mathrm{val}_v(\alpha^r))} & \text{if } v \notin S. \end{cases}$$

This measure has marginals ν'_v and $\nu_{\alpha^r,v}$. Hence, the quantity in (4.17) is an upper bound for the KR-distance $W(\nu'_v, \nu_{\alpha^r,v})$. It follows that we can choose α and r such that for every torsion point ω of $\mathbb{T}(\overline{\mathbb{K}})$, the point $p = \omega \cdot \alpha^r$ verifies for every v that

$$W(\nu'_v, \nu_{p,v}) < \frac{\varepsilon}{4\sum_{v'\in S} n_{v'}}$$

and $\sum_{v\notin S} n_v W(\nu'_v, \nu_{p,v}) < \varepsilon/4$. Hence $W_{\mathbb{K}}(\boldsymbol{\nu}', \boldsymbol{\nu}_p) < \varepsilon/2$ and thus $W_{\mathbb{K}}(\boldsymbol{\nu}, \boldsymbol{\nu}_p) < \varepsilon$.

Since the orbit of α^r under the action of the group of torsion points of $\mathbb{T}(\overline{\mathbb{K}})$ is Zariski dense, we have shown that, given $\varepsilon > 0$ and a nonempty open subset $U \subset X$, we can choose $p \in U(\overline{\mathbb{K}})$ satisfying

$$W_{\mathbb{K}}(\boldsymbol{\nu}, \boldsymbol{\nu}_p) < \varepsilon.$$

As in the proof of Proposition 2.5, let I be the set of hypersurfaces of X ordered by inclusion. For each $Y \in I$ choose a point $p_Y \in (X \setminus Y)(\overline{\mathbb{K}})$ such that

$$W_{\mathbb{K}}(\boldsymbol{\nu}, \boldsymbol{\nu}_{p_Y}) < \frac{1}{c(Y)}$$

with $c(Y)$ the number of components of Y. Thus, the net of algebraic points $(p_Y)_{Y\in I}$ is generic and the net of probability measures $(\boldsymbol{\nu}_{p_Y})_{Y\in I}$ converges to $\boldsymbol{\nu}$ in the KR-topology, proving the result. □

Proof of Theorem 4.7. Let $\boldsymbol{\nu} = (\nu_v)_v$ be a centered adelic measure on $N_{\mathbb{R}}$ such that each measure ν_v satisfies the condition (4.5). By Lemma 4.8, it satisfies

$$\eta_{\overline{D}}(\boldsymbol{\nu}) = \upmu^{\mathrm{ess}}_{\overline{D}}(X).$$

Proposition 4.11 implies that there is a generic net $(p_l)_{l\in I}$ of points in $\mathbb{T}(\overline{\mathbb{K}}) = X_0(\overline{\mathbb{K}})$ such that $(\boldsymbol{\nu}_{p_l})_{l\in I}$ converges to $\boldsymbol{\nu}$ with respect to the distance $W_{\mathbb{K}}$. On the other hand, by Lemma 4.8 we also have

$$\lim_l \mathrm{h}_{\overline{D}}(p_l) = \lim_l \eta_{\overline{D}}(\boldsymbol{\nu}_{p_l}) = \eta_{\overline{D}}(\boldsymbol{\nu}) = \upmu^{\mathrm{ess}}_{\overline{D}}(X),$$

and so the net $(p_l)_{l\in I}$ is $\overline{D}$-small. □

Corollary 4.12. *Let $v \in \mathfrak{M}_{\mathbb{K}}$. For every measure $\nu_v \in \mathcal{E}$ with $\operatorname{supp}(\nu_v) \subset F_v$ and $\mathrm{E}[\nu_v] \in B_v$, there is a generic $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X_0 such that the net of measures $(\nu_{p_l,v})_{l\in I}$ converges to ν_v with respect to the Kantorovich-Rubinstein distance. In particular, $(\nu_{p_l,v})_{l\in I}$ it also converges to ν_v in the weak-$*$ topology with respect to $\mathcal{C}_{\mathrm{b}}(N_{\mathbb{R}})$.*

Proof. Using the hypothesis that the point $u_v = \mathrm{E}[\nu_v]$ is in B_v, for each $v' \neq v$ we can find $u_{v'} \in B_{v'}$ such that $u_{v'} = 0$ except for a finite set $S' \subset \mathfrak{M}_{\mathbb{K}}$ and

$$\sum_{v'\in S'} n_{v'} u_{v'} = -n_v u_v.$$

For each $v' \neq v$, put $\nu_{v'} = \delta_{v'}$. The statement then follows from Theorem 4.7 applied to the centered adelic measure $\boldsymbol{\nu} = (\nu_v)_v$. □

Combining Theorems 4.3 and 4.7, we can obtain a criterion for when the direct image under the valuation map of the Galois orbits of a small net converges in the sense of measures. We show that in this case, the limit measure is concentrated in a single point.

Corollary 4.13. *Let $v \in \mathfrak{M}_{\mathbb{K}}$. The following conditions are equivalent:*

1. *for every $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X_0, the net of measures $(\nu_{p_l,v})_{l\in I}$ converges in the weak-$*$ topology with respect to $\mathcal{C}_{\mathrm{b}}(N_{\mathbb{R}})$;*

(2) *for every generic $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X_0, the net of measures $(\nu_{p_l,v})_{l\in I}$ converges in the weak-$*$ topology with respect to $\mathcal{C}_{\mathrm{c}}(N_{\mathbb{R}})$, the space of continuous functions on $N_{\mathbb{R}}$ with compact support;*

(3) *the face F_v contains only one point.*

When these equivalent conditions hold, the limit measures in (1) *and* (2) *coincide with the Dirac measure at the unique point of F_v.*

Proof. It is clear that (1) implies (2), and Theorem 4.3 shows that (3) implies (1). Now suppose that the face F_v has more than one point. Since F_v is the minimal face containing B_v, we can find distinct points $u_0, u_1, u_2 \in F_v$ such that

$$u_0 = \frac{u_1 + u_2}{2} \in B_v.$$

The probability measures δ_{u_0} and $\frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2}$ satisfy the conditions (4.5). By Corollary 4.12, we can find generic $\overline{D}$-small nets $(p_l)_{l\in I}$ and $(q_l)_{l\in I}$ such that the nets of measures $(\nu_{p_l,v})_{l\in I}$ and $(\nu_{q_l,v})_{l\in I}$ respectively converge to

$$\delta_{u_0} \quad \text{and} \quad \frac{1}{2}\delta_{u_1} + \frac{1}{2}\delta_{u_2}$$

in the KR-topology, and hence in the weak-$*$ topology with respect to $\mathcal{C}_{\mathrm{c}}(N_{\mathbb{R}})$. Combining these nets, we can obtain a net that does not converge in this weak-$*$ topology. Hence the condition (2) implies the condition (3).

The last statement follows from Theorem 4.3. □

When any of the equivalent conditions of Corollary 4.13 holds we say that the metrized divisor $\overline{D}$ satisfies the *modulus concentration property* at the place v. Thus Corollary 4.13 gives us a criterion for the modulus concentration property at a place. We next give a criterion for the modulus concentration property at all places simultaneously, that can be directly read from the roof function. Before giving it, we need some preliminary results and a definition.

Definition 4.14. A semipositive toric metrized $\mathbb{R}$-divisor $\overline{D}$ with D big is called *monocritical* if the minimum of $\eta_{\overline{D}}$ in $\mathcal{H}_{\mathbb{K}}$ is attained at a unique point. If this is the case, by Corollary 4.10, the minimum is attained at a point of $H_{\mathbb{K}}$. This point is called *the critical point* of $\overline{D}$.

Example 4.15. Let $\overline{D}^{\mathrm{can}}$ be a nef and big toric $\mathbb{R}$-divisor equipped with the canonical metric as in Example 4.1. Then all its local roof functions are zero. Taking a point x in the interior of the polytope, we have $\partial\vartheta_{\overline{D},v}(x) = \{0\}$ for every v. Hence $F_v = \{0\}$ for every v and $\overline{D}$ is monocritical with critical point $\mathbf{0} \in H_{\mathbb{K}}$.

Recall that $\Delta_{D,\max}$ denotes the convex set of points of Δ_D where $\vartheta_{\overline{D}}$ attains its maximum.

Proposition 4.16. *The following conditions are equivalent:*

(1) *the metrized $\mathbb{R}$-divisor $\overline{D}$ is monocritical;*

(2) *for every point $x \in \Delta_{D,\max}$, the set*

$$H_{\mathbb{K}} \cap \prod_{v\in\mathfrak{M}_{\mathbb{K}}} \partial\vartheta_{\overline{D},v}(x) \tag{4.18}$$

contains a unique element $\boldsymbol{u} = (u_v)_v \in H_{\mathbb{K}}$ and, for $v \in \mathfrak{M}_{\mathbb{K}}$, the point u_v is a vertex of $\partial\vartheta_{\overline{D},v}(x)$;

(3) *for every point $x \in \Delta_{D,\max}$, the point 0 is a vertex of $\partial\vartheta_{\overline{D}}(x)$;*

(4) *there exists a point $x \in \Delta_{D,\max}$ such that 0 is a vertex of $\partial\vartheta_{\overline{D}}(x)$;*

(5) *for all $v \in \mathfrak{M}_{\mathbb{K}}$, the set F_v contains only one point.*

When these equivalent conditions hold, $F_v = \{u_v\}$ *for every* v *and* $\boldsymbol{u}$ *is the critical point of* $\overline{D}$*.*

Proof. We prove first that (1) implies (2). Assume that $\overline{D}$ is monocritical. If the set (4.18) contains two elements $\boldsymbol{u}_i = (u_{i,v})_v \in H_{\mathbb{K}}$, $i = 1, 2$. Then the measures $\boldsymbol{\nu}_i = (\delta_{u_{i,v}})_v \in \mathcal{H}_{\mathbb{K}}$, $i = 1, 2$, satisfy that $\operatorname{supp}(\delta_{u_{i,v}}) \subset B_v$ for each v. In particular, $\operatorname{supp}(\delta_{u_{i,v}}) \subset F_v$ and $\mathrm{E}[\delta_{u_{i,v}}] \in B_v$. Thus by Lemma 4.8

$$\eta_{\overline{D}}(\boldsymbol{u}_1) = \eta_{\overline{D}}(\boldsymbol{u}_2) = \min_{\boldsymbol{\nu} \in \mathcal{H}_{\mathbb{K}}} \eta_{\overline{D}}(\boldsymbol{\nu})$$

contradicting the hypothesis that $\overline{D}$ is monocritical, and showing that (4.18) contains a unique element.

Assume now that the set (4.18) contains a single point $\boldsymbol{u} = (u_v)_v \in H_{\mathbb{K}}$ and there is a place $v_0 \in \mathfrak{M}_{\mathbb{K}}$ such that u_{v_0} is not a vertex of $\partial\vartheta_{\overline{D},v_0}(x)$. Then we can find two points $u_{v_0,1}, u_{v_0,2} \in \partial\vartheta_{\overline{D},v_0}(x)$ such that

$$u_{v_0} = \frac{u_{v_0,1} + u_{v_0,2}}{2}.$$

We consider the measure $\boldsymbol{\nu}_1 = (\delta_{u_v})_v$ and the measure $\boldsymbol{\nu}_2 = (\nu_v)_v$ defined by

$$\nu_v = \begin{cases} \delta_{u_v} & \text{if } v \neq v_0, \\ \dfrac{\delta_{u_{v_0,1}} + \delta_{u_{v_0,2}}}{2} & \text{if } v = v_0. \end{cases}$$

Then $\boldsymbol{\nu}_2$ is in (4.18) and, again by Lemma 4.8, we have that

$$\eta_{\overline{D}}(\boldsymbol{\nu}_1) = \eta_{\overline{D}}(\boldsymbol{\nu}_2) = \min_{\boldsymbol{\nu} \in \mathcal{H}_{\mathbb{K}}} \eta_{\overline{D}}(\boldsymbol{\nu})$$

contradicting the hypothesis that $\overline{D}$ is monocritical, and completing the proof of (2).

Assume that (2) is true and fix $x \in \Delta_{D,\max}$. Let $S \subset \mathfrak{M}_{\mathbb{K}}$ be the finite set of places where $u_v \neq 0$ or $\vartheta_{\overline{D},v}$ is not identically zero. We have that

$$\partial\vartheta_{\overline{D}}(x) = \sum_{v \in S} n_v \partial\vartheta_{\overline{D},v}(x).$$

Moreover, (2) implies that the equation

$$0 = \sum_{v \in S} n_v a_v \quad \text{with } a_v \in \partial\vartheta_{\overline{D},v}(x)$$

has a unique solution $a_v = u_v$ and this solution satisfies that a_v is a vertex of $\partial\vartheta_{\overline{D},v}(x)$. Therefore, by Lemma 3.15 we deduce that 0 is a vertex of $\partial\vartheta_{\overline{D}}(x)$. Hence (2) implies (3).

Since $\Delta_{D,\max}$ is nonempty, (3) implies (4).

Assume now that (4) is true. For each v, let $g_{1,v}$ and $g_{2,v}$ be the continuous concave functions on Δ_D in Notation 4.2. Since $\vartheta_{\overline{D}} = n_v g_{1,v} + n_v g_{2,v}$,

$$\partial\vartheta_{\overline{D}}(x) = n_v \partial g_{1,v}(x) + n_v \partial g_{2,v}(x).$$

Lemma 3.15 and the definition of the set B_v imply that this set contains one single point u_v, and that this point is a vertex of both $\partial g_{1,v}(x)$ and of $-\partial g_{2,v}(x)$. Hence B_v is already a face of $\partial g_{1,v}(x)$. Thus $F_v = B_v = \{u_v\}$ and so (4) implies (5).

By Lemma 4.8 it is clear that (5) implies (1) finishing the proof of the equivalence.

Assume now that $\overline{D}$ is monocritical. Since by Lemma 4.8 the point $\boldsymbol{u}$ in (2) satisfies that $\eta_{\overline{D}}(\boldsymbol{u}) = \min_{\boldsymbol{\nu} \in \mathcal{H}_{\mathbb{K}}} \eta_{\overline{D}}(\boldsymbol{\nu})$, it is the critical point. Following the proof of the equivalence we deduce that $F_v = \{u_v\}$ proving the last statement. □

For a given toric metrized $\mathbb{R}$-divisor, the condition of being monocritical and its critical point behave well with respect to scalar extensions. The following result

follows from the compatibility of toric metrics with scalar extensions in [BPS14a, Proposition 4.3.8].

Proposition 4.17. *Let X and $\overline{D}$ as before. Let $\mathbb{F} \subset \overline{\mathbb{K}}$ be a finite extension of $\mathbb{K}$ and write $\overline{D}_{\mathbb{F}}$ for the toric metrized $\mathbb{R}$-divisor on $X_{\mathbb{F}}$ obtained by scalar extension. If $\overline{D}$ is monocritical with critical point $(u_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$, then $\overline{D}_{\mathbb{F}}$ is also monocritical and its critical point $(u_w)_{w\in\mathfrak{M}_{\mathbb{F}}}$ is given by $u_w = u_v$ for all $v \in \mathfrak{M}_{\mathbb{K}}$ and w over v.*

We now give the criterion for modulus concentration at every place.

Theorem 4.18. *Let X and $\overline{D}$ be as before. The following conditions are equivalent:*

1. *for every $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X_0 and every place $v \in \mathfrak{M}_{\mathbb{K}}$, the net of measures $(\nu_{p_l,v})_{l\in I}$ converges.*
2. *the metrized $\mathbb{R}$-divisor $\overline{D}$ is monocritical;*

When these equivalent conditions hold,

$$\lim_{l\in I} \nu_{p_l,v} = \delta_{u_v},$$

where $(u_v)_v$ is the critical point of $\overline{D}$.

Proof. The theorem follows directly from Corollary 4.13 and Proposition 4.16. □

When there is modulus concentration for every place, we can show that the convergence holds not only in the weak-$*$ topology with respect to $\mathcal{C}_{\mathrm{b}}(N_{\mathbb{R}})$ but even in the stronger adelic KR-topology.

Theorem 4.19. *Let X and $\overline{D}$ be as before. Assume that $\overline{D}$ is monocritical. Let $\boldsymbol{u} = (u_v)_v$ be the critical point of $\overline{D}$ and set $\boldsymbol{\delta_u} = (\delta_{u_v})_v \in \mathcal{H}_{\mathbb{K}}$. Then, for every $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X_0, the net of centered adelic measures $(\boldsymbol{\nu}_{p_l})_{l\in I}$ converges to $\boldsymbol{\delta_u}$ in the adelic KR-topology. In particular, for every $v \in \mathfrak{M}_{\mathbb{K}}$, the net of measures $(\nu_{p_l,v})_{l\in I}$ converges to δ_{u_v} in the KR-topology.*

Proof. For each $v \in \mathfrak{M}_{\mathbb{K}}$, let $f_v \colon N_{\mathbb{R}} \to \mathbb{R}$ be the function given by

$$f_v(u) = \psi_{\overline{D},v}(u) - \Psi_D(u - u_v).$$

This is an adelic family of bounded continuous functions on $N_{\mathbb{R}}$ with $f_v = 0$ for all but a finite number of v. Consider then the function $\eta' \colon \mathcal{H}_{\mathbb{K}} \to \mathbb{R}$ given by

$$\eta'(\boldsymbol{\nu}) = \eta_{\overline{D}}(\boldsymbol{\nu}) + \sum_v n_v \int f_v \,\mathrm{d}\nu_v = -\sum_v n_v \int \Psi_D(u - u_v)\,\mathrm{d}\nu_v.$$

Since the net $(p_l)_{l\in I}$ is $\overline{D}$-small,

$$\lim_l \eta_{\overline{D}}(\boldsymbol{\nu}_{p_l}) = \lim_l \mathrm{h}_{\overline{D}}(p_l) = \upmu^{\mathrm{ess}}_{\overline{D}}(X).$$

By Theorem 4.18, the net of measures $(\nu_{p_l,v})_{l\in I}$ converges to δ_{u_v}. Using Corollary 4.10, we deduce that

$$\lim_l \eta'(\boldsymbol{\nu}_{p_l}) = \lim_l \eta_{\overline{D}}(\boldsymbol{\nu}_{p_l}) + \sum_v n_v \int f_v \,\mathrm{d}\delta_{u_v} = \upmu^{\mathrm{ess}}_{\overline{D}}(X) + \sum_v n_v f_v(u_v) = 0. \quad (4.19)$$

Choose a point x in the interior of Δ_D. Then there is a constant $c > 0$ such that, for all $u \in N_{\mathbb{R}}$,

$$\|u\| \le -c\,(\Psi_D - x)(u).$$

It follows from the definition of the Kantorovich-Rubinstein distance that, for each $v \in \mathfrak{M}_{\mathbb{K}}$,

$$W(\nu_{p_l,v}, \delta_{u_v}) \le \int \|u - u_v\|\,\mathrm{d}\nu_{p_l,v}(u).$$

Hence

$$W_{\mathbb{K}}(\boldsymbol{\nu}_{p_l}, \boldsymbol{\delta}_{\boldsymbol{u}}) \leq \sum_v n_v \int \|u - u_v\| \, \mathrm{d}\nu_{p_l,v}(u)$$
$$\leq -c \sum_v n_v \int (\Psi_D - x)(u - u_v) \, \mathrm{d}\nu_{p_l,v}(u) = c\,\eta'(\boldsymbol{\nu}_{p_l}),$$

where the last equality follows from the product formula in Proposition 2.1(2). By (4.19), this distance converges to 0, completing the proof. □

5. Equidistribution of Galois orbits and the Bogomolov property

We turn to the study of the limit measures of Galois orbits of $\overline{D}$-small nets of algebraic points in toric varieties. In this section, we denote by X a proper toric variety over a global field $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X with D big. For $v \in \mathfrak{M}_{\mathbb{K}}$, recall that $\mathrm{val}_v \colon \mathbb{T}_v^{\mathrm{an}} \to N_{\mathbb{R}}$ denotes the valuation map, defined in (4.1).

We first describe the limit measures in the monocritical case.

Definition 5.1. Given $v \in \mathfrak{M}_{\mathbb{K}}$ and $u \in N_{\mathbb{R}}$, the probability measure $\lambda_{\mathbb{S}_v,u}$ on X_v^{an} is defined as follows.

(1) When v is Archimedean, note that $\mathrm{val}_v^{-1}(u) = \mathbb{S}_v \cdot p$ for any point $p \in \mathrm{val}_v^{-1}(u)$ and where $\mathbb{S}_v = \mathrm{val}_v^{-1}(0) \simeq (S^1)^n$ is the compact torus of $\mathbb{T}_v^{\mathrm{an}}$. In this case, $\lambda_{\mathbb{S}_v,u}$ is the direct image under the translation by p of the Haar probability measure of $\mathbb{S}_v$.

(2) When v is non-Archimedean, consider the multiplicative seminorm on the group algebra $\mathbb{C}_v[M] \simeq \mathbb{C}_v[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ that, to a Laurent polynomial $\sum_{m \in M} \alpha_m \chi^m$, assigns the value $\max_m(|\alpha_m|_v \, \mathrm{e}^{-\langle m,u\rangle})$. This seminorm gives a point, denoted by $\theta(u)$, in the Berkovich space X_v^{an}. The point $\theta(u)$ is of type II or III in Berkovich's classification and lies in the preimage $\mathrm{val}_v^{-1}(u)$. We then set $\lambda_{\mathbb{S}_v,u} = \delta_{\theta(u)}$, the Dirac measure at this point.

The following result corresponds to Theorem 1.2 in the introduction, and shows that modulus concentration at every place implies the equidistribution property at every place. Due to the existing equidistribution theorems in the literature, we restrict its statement to divisors (rather than $\mathbb{R}$-divisors).

Theorem 5.2. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a semipositive toric metrized divisor on X with D big. The following conditions are equivalent:*

(1) *for every generic $\overline{D}$-small net $(p_l)_{l \in I}$ of algebraic points of X_0 and every place $v \in \mathfrak{M}_{\mathbb{K}}$, the net of probability measures $(\mu_{p_l,v})_{l \in I}$ on X_v^{an} converges;*

(2) *the metrized divisor $\overline{D}$ is monocritical.*

When these equivalent conditions hold, the limit measure in (1) *is $\lambda_{\mathbb{S}_v,u_v}$, with $u_v \in N_{\mathbb{R}}$ the v-adic component of the critical point of $\overline{D}$.*

The proof of Theorem 5.2 is done by reduction to the quasi-canonical case. The following is the characterization of quasi-canonical toric metrized $\mathbb{R}$-divisors in [BPS14b].

Proposition 5.3. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a semipositive toric metrized $\mathbb{R}$-divisor on X with D big. The following conditions are equivalent:*

(1) *$\overline{D}$ is quasi-canonical (Definition 2.7);*

(2) *$\vartheta_{\overline{D}}$ is constant;*

(3) *there is $\boldsymbol{u} = (u_v)_v \in H_{\mathbb{K}}$ and $(\gamma_v)_v \in \bigoplus_{v \in \mathfrak{M}_{\mathbb{K}}} \mathbb{R}$ such that*

$$\psi_{\overline{D},v}(u) = \Psi_D(u - u_v) - \gamma_v$$

for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $u \in N_{\mathbb{R}}$.

Proof. The equivalence of (1) and (3) is given by [BPS14b, Corollary 4.7]. The equivalence of (1) and (2) is given in the course of the proof of [BPS14b, Proposition 4.6], recalling that $\mathrm{vol}(D) = \deg_D(X)$ and noting that, since by assumption $\overline{D}$ is semipositive, $\widehat{\mathrm{vol}}_{\chi}(\overline{D}) = \mathrm{h}_{\overline{D}}(X)$. □

The following result gives the key step in the proof of Theorem 5.2.

Proposition 5.4. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a monocritical metrized $\mathbb{R}$-divisor on X with critical point $\boldsymbol{u} = (u_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$. Let $\overline{D}'$ be the toric metrized $\mathbb{R}$-divisor over D corresponding to the family of concave functions $\psi_{\overline{D}',v} \colon N_{\mathbb{R}} \to \mathbb{R}$, $v \in \mathfrak{M}_{\mathbb{K}}$, given by*

$$\psi_{\overline{D}',v}(u) = \Psi_D(u - u_v). \tag{5.1}$$

Then $\overline{D}'$ is quasi-canonical and every $\overline{D}$-small net of algebraic points of X_0 is also $\overline{D}'$-small.

Proof. The fact that $\overline{D}'$ is quasi-canonical is given by Proposition 5.3.

Let $(p_l)_{l\in I}$ be a $\overline{D}$-small net of algebraic points of X_0. By Theorem 4.19, the net of centered adelic measures $(\boldsymbol{\nu}_{p_l})_{l\in I}$ converges to $\boldsymbol{\delta}_{\boldsymbol{u}} = (\delta_{u_v})_v$ with respect to the adelic KR-distance. By Lemma 4.8, the function $\eta_{\overline{D}'}$ is continuous with respect to this distance. Using (4.9), we deduce that

$$\lim_l \mathrm{h}_{\overline{D}'}(p_l) = \lim_l \eta_{\overline{D}'}(\boldsymbol{\nu}_{p_l}) = \eta_{\overline{D}'}(\boldsymbol{\delta}_{\boldsymbol{u}}) = 0.$$

On the other hand, $\vartheta_{\overline{D}',v} = u_v$ for each v. Since the critical point $\boldsymbol{u}$ lies in the subspace $H_{\mathbb{K}}$, we have that $\vartheta_{\overline{D}'} = \sum_v n_v u_v = 0$. Hence,

$$\upmu^{\mathrm{ess}}_{\overline{D}'}(X) = \max_{x\in\Delta_D} \vartheta_{\overline{D}'}(x) = 0.$$

Thus $(p_l)_{l\in I}$ is $\overline{D}'$-small, as stated. □

Proof of Theorem 5.2. Suppose that the condition (1) holds. Given a generic $\overline{D}$-small net $(p_l)_{l\in I}$ of algebraic points of X_0 and $v \in \mathfrak{M}_{\mathbb{K}}$, the net of measures $(\mu_{p_l,v})_{l\in I}$ converges weakly with respect to the space $\mathcal{C}(X_v^{\mathrm{an}})$. Hence, the net of direct images $(\nu_{p_l,v})_{l\in I}$ converges weakly with respect to the space $\mathcal{C}_{\mathrm{c}}(N_{\mathbb{R}})$. By Corollary 4.13, for each v, the face F_v contains only one point. Proposition 4.16 then implies that $\overline{D}$ is monocritical, giving the condition (2).

Now suppose that the condition (2) holds. Let (Y, E) be the polarized toric variety associated to the polytope Δ_D. By the characterization of semipositive toric metrics in [BPS14a, Theorem 4.8.1], the metric in $\overline{D}$ induces a semipositive toric metric on E, and we denote by $\overline{E}$ the corresponding toric metrized divisor. We have that $\psi_{\overline{E},v} = \psi_{\overline{D},v}$ for all v, and so $\overline{E}$ is also monocritical with the same critical point as $\overline{D}$.

Let

$$\overline{E}' = (E, \|\cdot\|'_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$$

be the ample divisor E on Y equipped with the quasi-canonical toric metric given by Proposition 5.4, with $\overline{D}$ replaced by $\overline{E}$. Let $(p_l)_{l\in I}$ be a generic $\overline{D}$-small net of algebraic points of $X_0 = \mathbb{T} = Y_0$. It is also a generic $\overline{E}$-small net of algebraic points of Y_0. By Proposition 5.4 with $\overline{D}$ replaced by $\overline{E}$, it is also $\overline{E}'$-small.

By Theorem 2.11, for each place v the net $(\mu_{p_l,v})_{l\in I}$ converges to the normalized Monge-Ampère measure $\mu_v = \frac{1}{\deg_E(Y)} \mathrm{c}_1(E, \|\cdot\|'_v)^{\wedge n}$ on Y_v^{an}. Consider the real Monge-Ampère measure $\mathcal{M}(\psi_{\overline{E}',v})$ associated to the v-adic metric in $\overline{E}'$ as

in [BPS14a, Definition 2.7.1]. By the explicit formula (5.1) and [BPS14a, Example 2.7.5],

$$\mathcal{M}(\psi_{\overline{E}',v}) = \mathrm{vol}_M(\Delta_D)\delta_{u_v} = \frac{\deg_E(Y)}{n!}\delta_{u_v}.$$

Then [BPS14a, Theorem 4.8.11] implies that $\mu_v = \lambda_{\mathbb{S}_v,u_v}$. Therefore, the net of measures $(\mu_{p_l,v})_{l\in I}$ on X_v^{an} converges to $\lambda_{\mathbb{S}_v,u_v}$, giving the condition (1) and the last statement in the theorem. □

Example 5.5. Let $\overline{D}^{\mathrm{can}}$ be a big and nef toric divisor on X equipped with the canonical metric. Following Example 4.15, this toric metrized divisor is monocritical with critical point $\mathbf{0} \in H_{\mathbb{K}}$. Hence, it satisfies the v-adic equidistribution property with limit measure $\lambda_{\mathbb{S}_v,0}$, for every $v \in \mathfrak{M}_{\mathbb{K}}$.

In [Bil97], Bilu gave an equidistribution theorem for Galois orbits of sequences of points of small canonical height. This result is restricted to number fields and Archimedean places. However, and in contrast with the previous example, this result holds not just for generic, but for *strict* sequences of points, that is, sequences that eventually avoid any given proper torsion subvariety. This stronger version of the equidistribution property was used in a crucial way in *loc. cit.* to prove the Bogomolov property for the canonical height.

Here we extend this version of the equidistribution property to monocritical metrized $\mathbb{R}$-divisors on toric varieties (Theorem 5.7) and deduce from it the Bogomolov property (Theorem 1.5 in the introduction, or Theorem 5.12 below). Our proofs are similar to Bilu's and use Fourier analysis. Hence, for the rest of the section we restrict to the case when $\mathbb{K}$ is a number field and we only study the equidistribution over the Archimedean places. Following Remark 2.10, we restrict without loss of generality to sequences, instead of nets.

To formulate this extension, we have to modify slightly the notion of strict sequence. First we recall some standard terminology: a *subtorus* of $\mathbb{T}$ is an algebraic subgroup of $\mathbb{T}$ that is geometrically irreducible, a *translate of a subtorus* is a subvariety of $\mathbb{T}_{\overline{\mathbb{K}}}$ that is the orbit of a point $p \in \mathbb{T}(\overline{\mathbb{K}})$ by a subtorus, and a *torsion subvariety* is a translate of a subtorus by a torsion point of the group $\mathbb{T}(\overline{\mathbb{K}}) \simeq (\overline{\mathbb{K}}^{\times})^n$.

Definition 5.6. A sequence $(p_l)_{l\geq 1}$ of algebraic points of $\mathbb{T}$ is *strict* if, for every translate of a subtorus $U \subsetneq \mathbb{T}_{\overline{\mathbb{K}}}$, there is $l_0 \geq 1$ such that $p_l \notin U(\overline{\mathbb{K}})$ for all $l \geq l_0$. Equivalently, $(p_l)_{l\geq 1}$ is strict if, for every $m \in M \setminus \{0\}$ and every point $q \in X_0(\overline{\mathbb{K}})$, there is $l_0 \geq 1$ such that $\chi^m(p_l) \neq \chi^m(q)$ for all $l \geq l_0$.

Theorem 5.7. *Let X be a proper toric variety over a number field $\mathbb{K}$ and $\overline{D}$ a monocritical metrized $\mathbb{R}$-divisor on X. Then, for every strict $\overline{D}$-small sequence $(p_l)_{l\geq 1}$ of algebraic points of X_0 and every Archimedean place $v \in \mathfrak{M}_{\mathbb{K}}$, the sequence $(\mu_{p_l,v})_{l\geq 1}$ converges to the probability measure $\lambda_{\mathbb{S}_v,u_v}$, with $u_v \in N_{\mathbb{R}}$ the v-adic component of the critical point of $\overline{D}$.*

Proof. Let $(p_l)_{l\geq 1}$ be a strict $\overline{D}$-small sequence of algebraic points of X_0. For each $m \in M \setminus \{0\}$ consider the character

$$\chi^m \colon \mathbb{T} \longrightarrow \mathbb{G}_{\mathrm{m},\mathbb{K}}.$$

Since $(p_l)_{l\geq 1}$ is strict, the sequence $(\chi^m(p_l))_{l\geq 1}$ is generic.

We embed $\mathbb{G}_{\mathrm{m},\mathbb{K}} \hookrightarrow \mathbb{P}^1_{\mathbb{K}}$ as the principal open subset. Let $D_0 = \mathrm{div}(x_0)$ be the divisor at infinity on $\mathbb{P}^1_{\mathbb{K}}$, that we equip with the quasi-canonical toric metric corresponding to the adelic family of functions $\psi_{\overline{D}_0^m,v} \colon \mathbb{R} \to \mathbb{R}$ given by

$$\psi_{\overline{D}_0^m,v}(u) = \min(0, u - \langle m, u_v\rangle).$$

For each $v \in \mathfrak{M}_{\mathbb{K}}$, there is a commutative diagram

$$\begin{tikzcd} \mathbb{T}_v^{\mathrm{an}} \arrow[r, "\chi^m"] \arrow[d, "\mathrm{val}_v"'] & \mathbb{G}_{\mathrm{m},v}^{\mathrm{an}} \arrow[d, "\mathrm{val}_v"] \\ N_{\mathbb{R}} \arrow[r, "m"'] & \mathbb{R} \end{tikzcd}$$

The commutativity of this diagram implies that $\nu_{\chi^m(p_l),v} = m_*\nu_{p_l,v}$. By Theorem 4.19, the sequence $(\boldsymbol{\nu}_{p_l})_{l\geq 1}$ converges in the adelic KR-topology to the centered adelic measure $(\delta_{u_v})_v$ on $N_{\mathbb{R}}$. Hence, the sequence $(\boldsymbol{\nu}_{\chi^m(p_l)})_{l\geq 1}$ converges in the adelic KR-topology to the centered adelic measure $(\delta_{\langle m,u_v\rangle})_v$ on $\mathbb{R}$. By (4.9) and Lemma 4.8, the sequence of points $(\chi^m(p_l))_{l\geq 1}$ is $\overline{D}_0^m$-small.

Summarizing, the sequence $(\chi^m(p_l))_{l\geq 1}$ of algebraic points of $\mathbb{P}^1_0$ is generic and small with respect to the quasi-canonical toric metrized divisor $\overline{D}_0^m$. Theorem 2.11 then implies that the sequence of measures $(\mu_{\chi^m(p_l),v})_{l\geq 1}$ on the analytification $\mathbb{P}^{1,\mathrm{an}}_v \simeq \mathbb{P}^1(\mathbb{C})$ converges to $\lambda_{\mathbb{S}_v,\langle m,u_v\rangle}$.

Assume now that v is Archimedean. Since the space of probability measures on $X(\mathbb{C})$ is sequentially compact, by restricting to a subsequence we can suppose without loss of generality that $(\mu_{p_l,v})_{l\geq 1}$ converges to a measure μ. Since the sequence of direct images $((\mathrm{val}_v)_*\mu_{p_l,v})_{l\geq 1}$ converges in the KR-topology to the Dirac measure on the point $u_v \in N_{\mathbb{R}}$, we deduce that

$$\mathrm{supp}(\mu) \subset \mathrm{val}_v^{-1}(u_v) = \mathbb{S}_v \cdot \mathrm{e}^{-u_v}.$$

Let z be the standard affine coordinate of $\mathbb{P}^1(\mathbb{C})$. For each $m \in M \setminus \{0\}$, let z_m be a continuous function on $\mathbb{P}^1(\mathbb{C})$ that agrees with z on a neighborhood of $S^1 \cdot \chi^m(\mathrm{e}^{-u_v})$. Hence $(\chi^m)^*(z_m)$ agrees with the character χ^m on a neighborhood of $\mathbb{S}_v \cdot \mathrm{e}^{-u_v}$. Then

$$\begin{aligned}\int \chi^m \,\mathrm{d}\mu = \int (\chi^m)^*(z_m)\,\mathrm{d}\mu &= \lim_l \int (\chi^m)^*(z_m)\,\mathrm{d}\mu_{p_l,v} \\ &= \lim_l \int z_m \,\mathrm{d}(\chi^m)_*\mu_{p_l,v} = \lim_l \int z_m\,\mathrm{d}\mu_{\chi^m(p_l),v} \\ &= \int z_m\,\mathrm{d}\lambda_{S^1,\langle m,u_v\rangle} = \int z\,\mathrm{d}\lambda_{S^1,\langle m,u_v\rangle} = 0,\end{aligned}$$

where the last equality comes from Cauchy's formula. Hence $\int \chi^m\,\mathrm{d}\mu = 0$ for all $m \in M \setminus \{0\}$. By Fourier analysis, the only probability measure supported on $\mathbb{S}_v \cdot \mathrm{e}^{-u_v}$ satisfying this condition is $\lambda_{\mathbb{S}_v,u_v}$. Thus $\mu = \lambda_{\mathbb{S}_v,u_v}$, concluding the proof. □

Remark 5.8. Our notion of strict sequence is stronger than the one in [Bil97]. Nevertheless, for the canonical height on a projective space, a small sequence of points is strict in our sense if and only if it eventually avoids any fixed translate of a subtorus with essential minimum equal to 0. Such a translate of a subtorus is necessarily a torsion subvariety, see for instance Example 5.16. Hence, a small sequence of points that is strict in the sense of Bilu [Bil97] is also strict in the sense of Definition 5.6. Thus Theorem 5.7 applied to the canonically metrized divisor at infinity on a projective space specializes to [Bil97, Theorem 1.1].

Remark 5.9. To the best of our knowledge, even for the canonical metric it is still not know if the equidistribution property for strict sequences holds for the non-Archimedean places of a global field.

The toric Bogomolov conjecture can be stated as follows: let X be a toric variety and D an ample divisor on X. Let $V \subset X_{0,\overline{\mathbb{K}}}$ be a subvariety which is not torsion.

Then there exists $\varepsilon > 0$ such that the subset of algebraic points of V of canonical height bounded above by ε, is not dense in V. Equivalently, if $V \subset X_{0,\overline{\mathbb{K}}}$ is a subvariety such that $\mu^{\mathrm{ess}}_{\overline{D}^{\mathrm{can}}}(V) = 0$, then V is a torsion subvariety.

This conjecture was proved by Zhang in the number field case [Zha95]. Bilu obtained a proof of Zhang's theorem based on his equidistribution theorem. In what follows, we extend his approach to the general monocritical case over a number field.

Recall that X denotes a proper toric variety over a number field $\mathbb{K}$ and $\overline{D}$ a toric metrized $\mathbb{R}$-divisor on X. For a subset $V \subset X(\overline{\mathbb{K}})$, we denote by $\mu^{\mathrm{abs}}_{\overline{D}}(V)$ the absolute minimum of the height of its algebraic points. The fact that $\overline{D}$ is toric implies

$$\mu^{\mathrm{ess}}_{\overline{D}}(X) = \mu^{\mathrm{abs}}_{\overline{D}}(X_0), \tag{5.2}$$

see [BPS14b, Lemma 3.9(1)]. Therefore, for any subvariety $V \subset X_{0,\overline{\mathbb{K}}}$,

$$\mu^{\mathrm{ess}}_{\overline{D}}(V) \geq \mu^{\mathrm{abs}}_{\overline{D}}(V) \geq \mu^{\mathrm{abs}}_{\overline{D}}(X_0) = \mu^{\mathrm{ess}}_{\overline{D}}(X). \tag{5.3}$$

This motivates the following definition.

Definition 5.10. A subvariety $V \subset X_{0,\overline{\mathbb{K}}}$ is $\overline{D}$-*special* if

$$\mu^{\mathrm{ess}}_{\overline{D}}(V) = \mu^{\mathrm{ess}}_{\overline{D}}(X).$$

In particular, an algebraic point p of X_0 is $\overline{D}$-special if and only if $\mathrm{h}_{\overline{D}}(p) = \mu^{\mathrm{ess}}_{\overline{D}}(X)$.

We also propose the following terminology.

Definition 5.11. The toric metrized $\mathbb{R}$-divisor $\overline{D}$ satisfies the *Bogomolov property* if every $\overline{D}$-special subvariety of $X_{0,\overline{\mathbb{K}}}$ is a translate of a subtorus.

We consider the problem of deciding if a given toric metrized $\mathbb{R}$-divisor satisfies the Bogomolov property. The following result corresponds to Theorem 1.5 in the introduction, and shows that the answer is affirmative for monocritical metrics.

Theorem 5.12. *Let X be a proper toric variety over a number field $\mathbb{K}$ and $\overline{D}$ a monocritical metrized $\mathbb{R}$-divisor on X with critical point $\boldsymbol{u} = (u_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$. Let V be a $\overline{D}$-special subvariety of $X_{0,\overline{\mathbb{K}}}$. Then V is a translate of a subtorus.*

Furthermore, if $u_v \in \mathrm{val}_v(\mathbb{T}(\mathbb{K})) \otimes \mathbb{Q}$ for all v, then V is the translate of a subtorus by a $\overline{D}$-special point.

Before giving the proof of this theorem, we study special points and, more generally, special translates of subtori in the monocritical case. We first give a criterion for the existence of such points.

Proposition 5.13. *Let X be a proper toric variety over $\mathbb{K}$ and $\overline{D}$ a monocritical metrized $\mathbb{R}$-divisor on X with critical point $\boldsymbol{u} = (u_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$. Then there exists a $\overline{D}$-special point if and only if*

$$u_v \in \mathrm{val}_v(\mathbb{T}(\mathbb{K})) \otimes \mathbb{Q} \quad \textit{for all} \quad v \in \mathfrak{M}_{\mathbb{K}}. \tag{5.4}$$

If this is the case, then every $\overline{D}$-special point is of the form $q^{1/\ell}$ with $q \in X_0(\mathbb{K})$ and $\ell \geq 1$.

Proof. Suppose that there is a $\overline{D}$-special point $p \in X_0(\overline{\mathbb{K}})$. Choose a finite normal extension $\mathbb{F} \subset \overline{\mathbb{K}}$ of $\mathbb{K}$ where p is defined. Consider the norm of p relative to this extension, given by

$$\mathrm{N}^{\mathbb{F}}_{\mathbb{K}}(p) = \prod_{\tau\in\mathrm{Gal}(\mathbb{F}/\mathbb{K})} \tau\big(p^{[\mathbb{F}:\mathbb{K}]_i}\big)$$

where $\mathrm{Gal}(\mathbb{F}/\mathbb{K})$ and $[\mathbb{F} : \mathbb{K}]_i$ are the Galois group and the inseparable degree of the extension, respectively.

Let $v \in \mathfrak{M}_{\mathbb{K}}$. For every $\tau \in \mathrm{Gal}(\mathbb{F}/\mathbb{K})$, there is a place $w \in \mathfrak{M}_{\mathbb{F}}$ over v such that $\mathrm{val}_v(\tau(p)) = \mathrm{val}_w(p)$. By Corollary 4.9 and Proposition 4.17, we have that $\mathrm{val}_w(p) = u_v$ for any such place. It follows that $\mathrm{val}_v(\tau(p)) = u_v$ for all τ. Using that $\#\,\mathrm{Gal}(\mathbb{F}/\mathbb{K}) \cdot [\mathbb{F} : \mathbb{K}]_i = [\mathbb{F} : \mathbb{K}]$, we deduce that

$$\mathrm{val}_v(\mathrm{N}_{\mathbb{K}}^{\mathbb{F}}(p)) = \sum_{\tau} \mathrm{val}_v\big(\tau(p)^{[\mathbb{F}:\mathbb{K}]_i}\big) = [\mathbb{F} : \mathbb{K}]u_v.$$

Since $\mathrm{N}_{\mathbb{K}}^{\mathbb{F}}(p) \in \mathbb{T}(\mathbb{K})$, we get that $[\mathbb{F} : \mathbb{K}]u_v \in \mathrm{val}_v(\mathbb{T}(\mathbb{K}))$, proving the implication.

Conversely, assume that the condition (5.4) holds. Let $S \subset \mathfrak{M}_{\mathbb{K}}$ be a finite set containing the Archimedean places and those places v where $u_v \neq 0$. Set

$$\mathbb{T}(\mathbb{K})_S = \{p \in \mathbb{T}(\mathbb{K}) \mid \mathrm{val}_v(p) = 0 \text{ for all } v \notin S\}$$

and let $H_{\mathbb{K},S}$ be the subspace of $\bigoplus_{v\in S} N_{\mathbb{R}}$ defined by the equation $\sum_{v\in S} n_v z_v = 0$. Moreover, consider the lattice

$$\Gamma = H_{\mathbb{K},S} \cap \bigoplus_{v\in S} \mathrm{val}_v(\mathbb{T}(\mathbb{K}))$$

and the map $\mathrm{val}_S \colon \mathbb{T}(\mathbb{K})_S \to \Gamma$ given by $\mathrm{val}_S(p) = (\mathrm{val}_v(p))_{v\in S}$. By Dirichlet's unit theorem [Wei74, Chapter IV, §4, Theorem 9], the image Λ of this map is a sublattice that is commensurable to Γ. Thus $\Lambda \otimes \mathbb{Q} = \Gamma \otimes \mathbb{Q}$. Condition (5.4) implies that $(u_v)_{v\in S} \in \Gamma \otimes \mathbb{Q} = \Lambda \otimes \mathbb{Q}$. Hence, there is an integer $\ell \geq 1$ such that

$$(\ell u_v)_{v\in S} \in \Lambda.$$

In other terms, there is $q \in \mathbb{T}(\mathbb{K})_S$ such that $\mathrm{val}_v(q) = \ell u_v$ for all $v \in S$. By Corollary 4.9, the point $p = q^{1/\ell} \in \mathbb{T}(\overline{\mathbb{K}})$ is $\overline{D}$-special, proving the reverse implication.

To prove the last statement, suppose that the condition (5.4) holds and consider an arbitrary $\overline{D}$-special point $p' \in X_0(\overline{\mathbb{K}})$. Let p be the $\overline{D}$-special point constructed above and $\mathbb{F} \subset \overline{\mathbb{K}}$ a finite extension of $\mathbb{K}$ so that $p, p' \in \mathbb{T}(\mathbb{F})$. Then $\mathrm{val}_w(p'p^{-1}) = 0$ for all $w \in \mathfrak{M}_{\mathbb{F}}$. By Kronecker's theorem, the point $p'p^{-1}$ is torsion. We conclude that some positive power of p' lies in $\mathbb{T}(\mathbb{K})$, as stated. □

Next we characterize the translates of subtori that are $\overline{D}$-special. Let $U = T_{\overline{\mathbb{K}}} \cdot p$ be the translate of a subtorus $T \subset \mathbb{T}$ by a point $p \in X_0(\overline{\mathbb{K}})$. The subtorus T corresponds to a saturated sublattice Q of N; we denote by $\iota \colon Q \hookrightarrow N$ the corresponding inclusion map. Let $\mathbb{F} \subset \overline{\mathbb{K}}$ be a finite extension of $\mathbb{K}$ where p is defined. For each $w \in \mathfrak{M}_{\mathbb{F}}$, we consider the affine subspace of $N_{\mathbb{R}}$ given by

$$A_{U,w} = \mathrm{val}_w(p) + Q_{\mathbb{R}}.$$

Indeed $A_{U,w} = \mathrm{val}_w(U_w^{\mathrm{an}})$ and so this affine subspace depends only on U and not on a particular choice for the translating point p.

As explained in [BPS14a, §3.2], the normalization of the closure of U in $X_{\overline{\mathbb{K}}}$ can be given a structure of toric variety. Let Σ be the fan on $N_{\mathbb{R}}$ corresponding to X and Σ_Q the fan on $Q_{\mathbb{R}}$ obtained by restricting Σ to this latter linear space. Then the inclusion $\iota \colon Q_{\mathbb{R}} \hookrightarrow N_{\mathbb{R}}$ induces an equivariant map of toric varieties

$$\varphi_{p,\iota} \colon X_{\Sigma_Q,\overline{\mathbb{K}}} \to X_{\overline{\mathbb{K}}}$$

extending the inclusion $U \hookrightarrow \mathbb{T}_{\mathbb{K}}$.

Proposition 5.14. *Let X be a proper toric variety over a number field $\mathbb{K}$ and $\overline{D}$ a monocritical metrized $\mathbb{R}$-divisor on X with critical point $\boldsymbol{u} = (u_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$. Let $U = T_{\overline{\mathbb{K}}} \cdot p \subset X_{0,\overline{\mathbb{K}}}$ be the translate of a subtorus $T \subset \mathbb{T}$ by a point $p \in X_0(\overline{\mathbb{K}})$ defined over a finite extension $\mathbb{F} \subset \overline{\mathbb{K}}$ of $\mathbb{K}$. For a place w in $\mathfrak{M}_{\mathbb{F}}$ denote by $v(w)$ the place in $\mathfrak{M}_{\mathbb{K}}$ below w. Then we have the following properties.*

(1) *The translate U is $\overline{D}$-special if and only if $u_{v(w)} \in A_{U,w}$ for all $w \in \mathfrak{M}_{\mathbb{F}}$.*

(2) *If the translate U is $\overline{D}$-special, then the metrized $\mathbb{R}$-divisor $\varphi_{p,\iota}^{*}\overline{D}$ is monocritical and its critical point is $(u_{v(w)} - \mathrm{val}_w(p))_{w\in\mathfrak{M}_{\mathbb{F}}}$.*

Proof. By passing to a suitable large finite extension of $\mathbb{K}$ and applying Proposition 4.17, we can reduce to the case when U is the translate of a $\mathbb{K}$-rational point, that is, $U = T_{\overline{\mathbb{K}}} \cdot p$ with $p \in X_0(\mathbb{K})$. With this assumption, $\mathbb{F} = \mathbb{K}$ and we set $v \coloneqq w = v(w)$.

Since $\overline{D}$ is a semipositive toric metrized divisor with D big, the virtual support function Ψ_D is concave and its associated polytope has dimension n. Hence, there is $m \in M_{\mathbb{R}}$ such that $\langle m, u\rangle > \Psi_D(u)$ for all $u \neq 0$. Moreover, the metric functions $\psi_{\overline{D},v}$ are concave for all $v \in \mathfrak{M}_{\mathbb{K}}$.

Consider the toric metrized $\mathbb{R}$-divisor $\overline{E} \coloneqq \varphi_{\iota,p}^{*}\overline{D}$ on the toric variety X_{Σ_Q}. By [BPS14a, Proposition 4.3.19], its virtual support function and metric functions are given, for $z \in Q_{\mathbb{R}}$, by

$$\Psi_E(z) = \Psi_D(\iota(z)), \qquad \psi_{\overline{E},v}(z) = \psi_{\overline{D},v}(\mathrm{val}_v(p) + \iota(z)).$$

Therefore Ψ_E is concave and satisfies $\langle \iota^{\vee} m, z\rangle > \Psi_E(z)$ for all $z \in Q_{\mathbb{R}} \setminus \{0\}$. Hence, the $\mathbb{R}$-divisor E is big. Moreover, the metric functions $\psi_{\overline{E},v}$ are concave and so $\overline{E}$ is semipositive.

Since U is identified with a dense open subset of $X_{\Sigma_Q,\overline{\mathbb{K}}}$, we have

$$\upmu^{\mathrm{ess}}_{\overline{D}}(U) = \upmu^{\mathrm{ess}}_{\overline{E}}(X_{\Sigma_Q}).$$

Consider the affine subspace $\boldsymbol{A}_U = \bigoplus_v A_{U,v}$ of $\bigoplus_v N_{\mathbb{R}}$. By Corollary 4.10,

$$\upmu^{\mathrm{ess}}_{\overline{E}}(X_{\Sigma_Q}) = \min_{\boldsymbol{u}'\in H_{\mathbb{F}}\cap \boldsymbol{A}_U} \sum_v -n_v \psi_{\overline{D},v}(u'_v), \qquad \upmu^{\mathrm{ess}}_{\overline{D}}(X) = \min_{\boldsymbol{u}'\in H_{\mathbb{F}}} \sum_v -n_v \psi_{\overline{D},v}(u'_v).$$

Since $\overline{D}$ monocritical, the minimum in the right equality is attained only at $\boldsymbol{u}' = (u_v)_v$. We conclude that $\upmu^{\mathrm{ess}}_{\overline{E}}(U) = \upmu^{\mathrm{ess}}_{\overline{D}}(X)$ if and only if $u_v \in A_{U,v}$ for all $v \in \mathfrak{M}_{\mathbb{K}}$, proving both statements. □

Corollary 5.15. *Let X be a proper toric variety over a number field $\mathbb{K}$ and $\overline{D}$ a monocritical metrized $\mathbb{R}$-divisor on X with critical point $\boldsymbol{u} = (u_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$, and suppose that $u_v \in \mathrm{val}_v(\mathbb{T}(\mathbb{K})) \otimes \mathbb{Q}$ for all $v \in \mathfrak{M}_{\mathbb{K}}$. Then a translate of a subtorus of X_0 is $\overline{D}$-special if and only if it is the translate of a subtorus by a $\overline{D}$-special point.*

Proof. Clearly, the translate of a subtorus by a $\overline{D}$-special point is $\overline{D}$-special. To prove the reverse implication, let U be a $\overline{D}$-special translate of a subtorus and write $U = T_{\overline{\mathbb{K}}} \cdot p$ as in the statement of Proposition 5.14. By this result, the toric metrized $\mathbb{R}$-divisor $\overline{E} = \varphi_{p,\iota}^{*}\overline{D}$ is monocritical and, for each $v \in \mathfrak{M}_{\mathbb{K}}$ and $w \in \mathfrak{M}_{\mathbb{F}}$ over v,

$$u_v \in A_{U,w} \cap \mathrm{val}_v(\mathbb{T}(\mathbb{K})) \otimes \mathbb{Q} \subset A_{U,w} \cap \mathrm{val}_w(\mathbb{T}(\mathbb{F})) \otimes \mathbb{Q}.$$

Since $p \in X_0(\mathbb{F})$,

$$A_{U,w} \cap \mathrm{val}_w(\mathbb{T}(\mathbb{F})) \otimes \mathbb{Q} = \mathrm{val}_w(p) + \mathrm{val}_w(T(\mathbb{F})) \otimes \mathbb{Q}.$$

Hence $u_v - \mathrm{val}_w(p) \in \mathrm{val}_w(T(\mathbb{F})) \otimes \mathbb{Q}$. Extending the base field to $\mathbb{F}$ and restricting to X_{Σ_Q}, Proposition 5.13 implies that this toric variety contains an $\overline{E}$-special point. Hence U contains a $\overline{D}$-special point and it is the translate of T by this point, as stated. □

Example 5.16. Let $\overline{D}^{\mathrm{can}}$ be a nef and big toric $\mathbb{R}$-divisor on the proper toric variety X, equipped with the canonical metric. By Example 4.15, it is monocritical with critical point $\boldsymbol{0} \in H_{\mathbb{K}}$. Hence, $p \in X_0(\overline{\mathbb{K}})$ is $\overline{D}^{\mathrm{can}}$-special if and only if $\mathrm{val}_v(p) = 0$ for every $v \in \mathfrak{M}_{\mathbb{K}}$. By Kronecker's theorem, this is also equivalent to the fact that p is torsion. Hence, Corollary 5.15 shows that a translate of a subtorus that

is $\overline{D}^{\mathrm{can}}$-special is necessarily the translate of a subtorus by a torsion point, that is, a torsion subvariety.

Proof of Theorem 5.12. Let $U \subset X_{0,\overline{\mathbb{K}}}$ be the minimal translate of a subtorus containing the subvariety V and let Q and Σ_Q be defined before Proposition 5.14. By (5.2) and (5.3), we have $\upmu^{\mathrm{abs}}_{\overline{D}}(U) = \upmu^{\mathrm{ess}}_{\overline{D}}(U)$ and

$$\upmu^{\mathrm{ess}}_{\overline{D}}(X) = \upmu^{\mathrm{abs}}_{\overline{D}}(X_0) \leq \upmu^{\mathrm{abs}}_{\overline{D}}(U) \leq \upmu^{\mathrm{abs}}_{\overline{D}}(V) \leq \upmu^{\mathrm{ess}}_{\overline{D}}(V) = \upmu^{\mathrm{ess}}_{\overline{D}}(X).$$

Therefore, U is $\overline{D}$-special. By Proposition 5.14(2), $\overline{D}$ pulls back to a monocritical metrized $\mathbb{R}$-divisor on X_{Σ_Q}, the normalization of the closure of U in $X_{\overline{\mathbb{K}}}$. Replacing X by this toric variety, we reduce to the case where $U = X_{0,\overline{\mathbb{K}}}$.

Using Proposition 2.5, we choose a sequence $(p_l)_{l\geq 1}$ of algebraic points of V, that is generic in V and satisfies

$$\lim_{l} \mathrm{h}_{\overline{D}}(p_l) = \upmu^{\mathrm{ess}}_{\overline{D}}(V).$$

Since V is not contained in any proper translate of a subtorus, this sequence is strict and, since V is $\overline{D}$-special, it is also $\overline{D}$-small.

Applying Theorem 5.7 to an Archimedean place $v \in \mathfrak{M}_{\mathbb{K}}$, we obtain that the sequence of measures $(\mu_{p_l,v})_{l\geq 1}$ converges to a measure whose support is the translate $\mathbb{S}_v \cdot \mathrm{e}^{-u_v}$ of the compact subtorus, with u_v the v-adic coordinate of the critical point of $\overline{D}$. Hence $\mathbb{S}_v \cdot \mathrm{e}^{-u_v} \subset V^{\mathrm{an}}_v$. Since $\mathbb{S}_v \cdot \mathrm{e}^{-u_v}$ is dense in X^{an}_v with respect to the Zariski topology, it follows that $V = X_{0,\overline{\mathbb{K}}}$, proving the result. □

By Theorem 5.12 and Example 5.16, the canonical toric metrized $\mathbb{R}$-divisor $\overline{D}^{\mathrm{can}}$ satisfies the Bogomolov property, and every $\overline{D}^{\mathrm{can}}$-special subvariety is torsion. Hence, Theorem 5.12 extends Zhang's theorem to the general monocritical case. On the other hand, in §6.3 we will give examples of non-monocritical metrized divisors not satisfying the Bogomolov property.

6. Examples

The obtained criteria can be applied in concrete situations, to decide if a given semipositive toric metrized $\mathbb{R}$-divisor satisfies properties like modulus concentration or equidistribution. In this section, we consider translates of subtori with the canonical height, and toric metrized $\mathbb{R}$-divisors equipped with positive smooth metrics at the Archimedean places and canonical metrics at the non-Archimedean ones. We also give a family of counterexamples to the Bogomolov property in the non-monocritical case.

6.1. **Translates of subtori with the canonical height.** Let X a proper toric variety of dimension n over a global field $\mathbb{K}$ and D a big and nef toric $\mathbb{R}$-divisor on X. Let Ψ_D be its virtual support function.

We denote by $\overline{D}^{\mathrm{can}}$ this $\mathbb{R}$-divisor equipped with the canonical metric as in Example 4.1. This toric metrized $\mathbb{R}$-divisor satisfies that, for all $v \in \mathfrak{M}_{\mathbb{K}}$,

$$\psi_{\overline{D}^{\mathrm{can}},v} = \Psi_D \quad \text{and} \quad \vartheta_{\overline{D}^{\mathrm{can}},v} = 0.$$

Since D is big, Δ_D has dimension n. Every point x in the interior of Δ_D maximizes the global roof function and $\partial\vartheta_{\overline{D}^{\mathrm{can}},v}(x) = \{0\}$. Therefore, for all $v \in \mathfrak{M}_{\mathbb{K}}$,

$$B_v = \{0\} \quad \text{and} \quad F_v = \{0\}.$$

By Proposition 4.16, the canonical metric is monocritical and so, by Theorem 5.2, $\overline{D}^{\mathrm{can}}$ satisfies the equidistribution property at every place (Example 5.5).

We next study the toric metrics on D that are obtained as inverse image by an equivariant map of a canonical metrized toric divisor on an projective space.

Let $v \in \mathfrak{M}_{\mathbb{K}}$. If v is Archimedean, we set $\lambda_v = 1$ whereas, if v is non-Archimedean, we set λ_v as the positive generator of the discrete subgroup $\mathrm{val}_v(\mathbb{K}^\times)$ of $\mathbb{R}$. A piecewise affine function is said to be *λ_v-rational* if all its defining affine functions $\langle x, u\rangle + b$ satisfy $x \in M_{\mathbb{Q}}$ and $b \in \lambda_v\mathbb{Q}$.

Let $\mathbb{P}^r_{\mathbb{K}}$ be a standard projective space over $\mathbb{K}$ with homogeneous coordinates $(z_0 : \cdots : z_r)$ and H the hyperplane at infinity, defined by the equation $z_0 = 0$. Denote by $\overline{H}^{\mathrm{can}}$ this toric divisor equipped with the canonical metric. As seen in [BPS14a, Example 3.7.11], if $\psi\colon N_{\mathbb{R}} \to \mathbb{R}$ is a concave λ_v-rational piecewise affine function with $|\psi - \Psi_D|$ is bounded, then there is an integer $r > 0$ and a toric morphism $\iota\colon X \to \mathbb{P}^r_{\mathbb{K}}$ such that

$$\psi = \psi_{\iota^*\overline{H}^{\mathrm{can}},v}.$$

Hence, any such function ψ can be realized as the v-adic metric function of a toric metrized divisor on D. This allows us to construct many examples, both monocritical and non-monocritical, of metrized toric divisors.

In the next examples, we fix $\mathbb{K} = \mathbb{Q}$ and, as before, we denote by $\overline{H}^{\mathrm{can}}$ the hyperplane at infinity with the canonical metric.

Example 6.1. Let $\iota\colon \mathbb{G}_{\mathrm{m},\mathbb{Q}} \to \mathbb{P}^2_{\mathbb{Q}}$ be the map given by

$$\iota(t) = (1 : t/2 : t).$$

Let X the normalization of the closure of $\iota(\mathbb{G}_{\mathrm{m},\mathbb{Q}})$ and $\overline{D} = \iota^*(\overline{H}^{\mathrm{can}})$. Then $X = \mathbb{P}^1_{\mathbb{Q}}$ and D is the divisor at infinity.

We have $\Delta_D = [0,1]$. As explained in [BPS14a, Example 5.1.16], for each $v \in \mathfrak{M}_{\mathbb{Q}}$ the graph of the local roof function associated to $\overline{D}$ is given by the upper envelope of the extended polytope

$$\mathrm{conv}((0,0), (1, \log|1/2|_v), (1, \log|1|_v)) \subset \mathbb{R}\times\mathbb{R}.$$

The graphs of these functions are represented in Figure 1. Thus, for $x \in [0,1]$

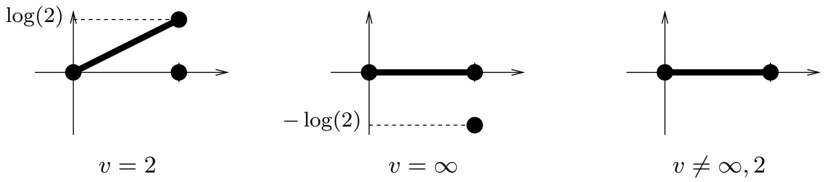

FIGURE 1. Local roof functions in Example 6.1

we have $\vartheta_2(x) = x\log(2)$ and $\vartheta_v(x) = 0$ for $v \neq 2$. The global roof function is $\vartheta(x) = x\log(2)$ and the only point that maximizes it is $x = 1$. Moreover, $\partial\vartheta_2(1) = (-\infty, \log(2)]$ and $\partial\vartheta_v(1) = (-\infty, 0]$ for $v \neq 2$. With Notation 4.2, we have

$$B_2 = [0, \log(2)], \qquad F_2 = [-\infty, \log(2)],$$
$$B_v = [-\log(2), 0], \quad F_v = [-\infty, 0] \quad \text{for } v \neq 2.$$

By Corollary 4.13, this metrized divisor does not satisfy the modulus concentration property at any place. *A fortiori*, it does not satisfy the equidistribution property at any place.

Indeed, by (4.3) we have $\mu^{\mathrm{ess}}_{\overline{D}}(X) = \log(2)$. Let $(\omega_l)_{l\geq 1}$ be a sequence given by a choice of a primitive l-th root of the unity, $a \neq 2$ a positive prime number and r an integer with $\log(a) \leq r\log(2)$. Choose any r-th root of a and consider the generic sequences of points

$$p_l = (1 : \omega_l) \quad \text{and} \quad q_l = (1 : 2a^{-1/r}\omega_l) \quad \text{for } l \geq 1.$$

For every $v \in \mathfrak{M}_{\mathbb{Q}}$, $l \geq 1$, $p \in \mathrm{Gal}(p_l)_v$ and $q \in \mathrm{Gal}(q_l)_v$ we have $(\mathrm{val}_v)_*(p) = 0$ and

$$(\mathrm{val}_v)_*(q) = \begin{cases} \log(2) & \text{if } v = 2, \\ \frac{-1}{r}\log(a) & \text{if } v = a, \\ -\log(2) + \frac{1}{r}\log(a) & \text{if } v = \infty, \\ 0 & \text{if } v \neq 2, a, \infty. \end{cases}$$

Either by computing the local roof functions of $\overline{D}$ or the Weil height of the image of these points under the inclusion ι, we deduce that

$$\mathrm{h}_{\overline{D}}(p_l) = \log(2) \quad \text{ and } \quad \mathrm{h}_{\overline{D}}(q_l) = \log(2).$$

Therefore both sequences are $\overline{D}$-small. For any place v, the sequence $\mu_{p_l,v}$ converges to $\lambda_{\mathbb{S}_v,0}$. By contrast, if we denote $u_v = (\mathrm{val}_v)_*(q)$ for any $q \in \mathrm{Gal}(q_l)_v$, then $\mu_{q_l,v}$ converges to $\lambda_{\mathbb{S}_v,u_v}$. This shows that neither the modulus concentration nor the equidistribution properties hold for the places $2, a, \infty$. Varying a, we deduce that these properties do not hold at any place of $\mathbb{Q}$.

The metric of $\overline{D}$ at the Archimedean place is the canonical one. The metrics at the non-Archimedean places can be interpreted in terms of integral models. Let $\mathcal{X}$ be the blow up of $\mathbb{P}^1_{\mathbb{Z}}$ at the point $(1 : 0)$ over the prime 2. The fibre of the structural map $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ over the point 2 has two components: the exceptional divisor of the blow up, that we denote by E, and the strict transform of the fibre of $\mathbb{P}^1_{\mathbb{Z}}$, that we denote Y. Consider the divisor

$$\mathcal{D} = \overline{\infty} + Y,$$

where $\overline{\infty}$ denotes the closure in $\mathcal{X}$ of the point $(0 : 1) \in \mathbb{P}^1(\mathbb{Q})$. The pair $(\mathcal{X}, \mathcal{D})$ is a model of (X, D). For each non-Archimedean place v, this model induces an algebraic metric on D that agrees with the v-adic metric of $\overline{D}$.

Example 6.2. Consider now the map $\iota\colon \mathbb{G}_{\mathrm{m},\mathbb{Q}} \to \mathbb{P}^2_{\mathbb{Q}}$ given by

$$\iota(t) = (t^{-1} : 1/2 : t).$$

Let X be the normalization of the closure of $\iota(\mathbb{G}_{\mathrm{m},\mathbb{Q}})$ and $\overline{D} = \iota^*(\overline{H}^{\mathrm{can}})$. In this case, $X = \mathbb{P}^1_{\mathbb{Q}}$ and D is the divisor at infinity plus the divisor at zero.

We have $\Delta_D = [-1, 1]$. As before, we compute the local roof functions using [BPS14a, Example 5.1.16]. Their graphs are represented in Figure 2. For $x \in [0, 2]$,

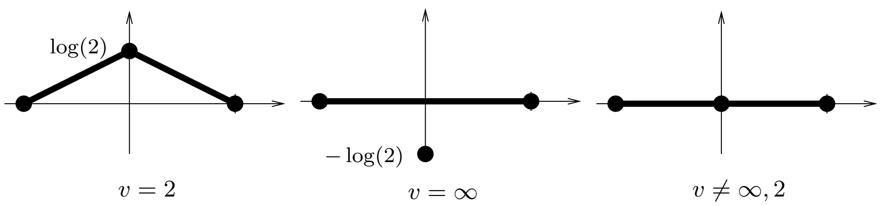

FIGURE 2. Local roof functions in Example 6.2

we have $\vartheta_2(x) = (1 - |x|)\log(2)$ and $\vartheta_v(x) = 0$ for $v \neq 2$. Thus, the global roof function is $\vartheta(x) = (1 - |x|)\log(2)$. Its maximum is attained only at the point $x = 0$. In this case, $\partial\vartheta_2(0) = [-\log(2), \log(2)]$ and $\partial\vartheta_v(0) = \{0\}$ for $v \neq 2$. We deduce that

$$B_2 = \{0\},\ F_2 = [-\log(2), \log(2)] \quad \text{ and } \quad B_v = \{0\},\ F_v = \{0\} \text{ for } v \neq 2. \tag{6.1}$$

By Corollary 4.13, $\overline{D}$ satisfies modulus concentration for all places except the place 2. This toric metrized divisor is not monocritical, and so we cannot apply Theorem 5.2 in this case. Indeed, later we will see that $\overline{D}$ does not satisfy the equidistribution property at any other place of $\mathbb{Q}$ (Example 7.6).

As in the previous example, the metric of $\overline{D}$ at the Archimedean place is the canonical one, and those at the non-Archimedean places can be interpreted in terms of integral models. Let $\mathcal{X}$ be the blow up of $\mathbb{P}^1_{\mathbb{Z}}$ at the points $(1:0)$ and $(0:1)$ over the prime 2. The fibre of the structural map $\mathcal{X} \to \operatorname{Spec}(\mathbb{Z})$ over the point 2 has three components. Consider the divisor

$$\mathcal{D} = \overline{\infty} + \overline{0},$$

where $\overline{\infty}$ denotes the closure in $\mathcal{X}$ of the point $(0:1) \in \mathbb{P}^1(\mathbb{Q})$ and $\overline{0}$ the closure of the point $(1:0)$. The pair $(\mathcal{X}, \mathcal{D})$ is a model of (X, D). For each non-Archimedean place v, this model induces an algebraic metric on D that agrees with the v-adic metric of $\overline{D}$.

Example 6.3. This time we consider the map $\iota\colon \mathbb{G}_{\mathrm{m},\mathbb{Q}} \to \mathbb{P}^3_{\mathbb{Q}}$ given by

$$\iota(t) = (1 : t/2 : t^2/2 : t^3).$$

Let X be the normalization of the closure of $\iota(\mathbb{G}_{\mathrm{m},\mathbb{Q}})$ and $\overline{D} = \iota^*(\overline{H}^{\mathrm{can}})$. In this case, $X = \mathbb{P}^1_{\mathbb{Q}}$ and D is three times the divisor at infinity.

We have $\Delta_D = [0, 3]$ and the local roof functions are represented in Figure 3. They are given by $\vartheta_2(x) = \log(2)\min(x, 1, 3 - x)$ and $\vartheta_v(x) = 0$ for $v \neq 2$. The

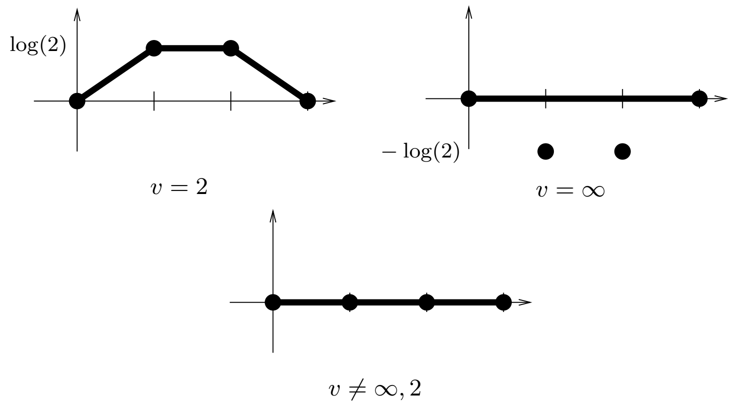

FIGURE 3. Local roof functions in Example 6.2

global roof function is thus $\vartheta(x) = \log(2)\min(x, 1, 3 - x)$, which is maximized at any point of the interval $[1, 2]$. Choosing the maximizing point $x = 3/2$, we have $\partial\vartheta_v(3/2) = \{0\}$ for all v.

Thus $\overline{D}$ is monocritical, by Proposition 4.16. By Corollary 4.13 and Theorem 5.2, it satisfies both the modulus concentration and the equidistribution properties for any place. By Theorem 5.12, it also satisfies the Bogomolov property.

6.2. **Positive Archimedean metrics.** The following result covers many of the examples considered in [BPS14a, BMPS12, BPS14b]: twisted Fubiny-Study metrics on projective spaces, metrics from polytopes, Fubini-Study metrics on toric bundles, ℓ^p-metrics on toric varieties, and Fubini-Study metrics on weighted projective spaces. All of them consist of toric varieties over $\mathbb{Q}$ with a toric divisor equipped with a positive smooth metric at the Archimedean place and the canonical metric at the non-Archimedean ones.

Theorem 6.4. *Let X a proper toric variety over a number field $\mathbb{K}$ and $\overline{D} = (D, (\|\cdot\|_v)_{v\in\mathfrak{M}_\mathbb{K}})$ a semipositive toric metrized $\mathbb{R}$-divisor with D big. We assume that, when v is Archimedean, $\|\cdot\|_v$ is a positive smooth metric on the principal open subset $X_{0,v}^{\mathrm{an}}$ whereas, when v is non-Archimedean, it is the v-adic canonical metric of D. Then $\overline{D}$ is monocritical. In particular, it satisfies the equidistribution property for every place of $\mathbb{K}$.*

When $\mathbb{K} = \mathbb{Q}$, the v-adic limit measure is $\lambda_{\mathbb{S}_v,0}$ for every $v \in \mathfrak{M}_\mathbb{Q}$.

Proof. Since the metric is smooth and positive for v Archimedean, the proof of [BPS14a, Proposition 4.4.1] implies that the metric function $\psi_{\overline{D},v}$ is smooth and strictly concave, in the sense that its Hessian is negative definite. Therefore $\psi_{\overline{D},v}$ is of Legendre type in the sense of [BPS14a, Definition 2.4.1] and, by [BPS14a, Theorem 2.4.2(2)], the local roof function $\vartheta_{\overline{D},v}$ is of Legendre type. In particular, $\vartheta_{\overline{D},v}$ is smooth and strictly concave on the interior of Δ_D and the sup-differential at any point of the border of the polytope is empty.

For the non-Archimedean places, the metrics are canonical and so their local roof functions are zero. Hence

$$\vartheta_{\overline{D}} = \sum_{v|\infty} n_v \vartheta_{\overline{D},v},$$

this function is smooth and strictly concave on the interior of Δ_D, and its sup-differential at any point of the border of Δ_D is empty. This implies that there is a unique maximizing point $x_{\max} \in \Delta_D$, that it lies in the interior of the polytope, and that $\partial\vartheta_{\overline{D}}(x_{\max}) = \{0\}$.

The first assertion then follows from Proposition 4.16. The rest of the statement follows from Theorem 5.2. □

Example 6.5. Let $X = \mathbb{P}^1_\mathbb{Q}$ and $\overline{D}$ the divisor at infinity equipped with the Fubini-Study metric at the Archimedean place and the canonical metric at the non-Archimedean ones. By Theorem 6.4, this toric metrized divisor satisfies the equidistribution property at every place. Moreover, the limit measure of the Galois orbits of any generic $\overline{D}$-small sequence is $\lambda_{\mathbb{S}_v,0}$.

Recall that the canonical metric at the non-Archimedean places corresponds to the canonical model of $(\mathbb{P}^1_\mathbb{Q}, \infty)$ given by $(\mathbb{P}^1_\mathbb{Z}, \overline{\infty})$, where $\overline{\infty}$ is the closure of the point $(0:1) \in \mathbb{P}^1(\mathbb{Q})$. If we change the integral model, different phenomena may occur. For instance, if we consider the integral model of Example 6.1, then the maximum of the global roof function is attained at the interior of the polytope. Since the global roof function is differentiable in the interior of the polytope, we deduce that the sup-differential is reduced to one point. By Proposition 4.16, this new toric metrized divisor is also monocritical.

By contrast, if we consider the divisor $D' = 0 + \infty$ with the Fubini-Study metric at the Archimedean place and the metrics induced by the integral model of Example 6.2, then the maximum of the global roof function is attained at the point zero and the sup-differential at this point is $[-\log(2), \log(2)]$. Since zero is not a vertex of this set, by Proposition 4.16 this divisor is not monocritical. Hence it does not satisfy the equidistribution property at the Archimedean place.

6.3. **Counterexamples to the Bogomolov property.** In this section, we give examples of toric metrized divisors not satisfying the Bogomolov property. For simplicity, we restrict to the case $\mathbb{K} = \mathbb{Q}$. As in §6.1, we denote by $\overline{H}^{\mathrm{can}}$ the canonical metrized divisor at infinity on a projective space.

Example 6.6. Consider the map $\iota\colon \mathbb{G}_{\mathrm{m},\mathbb{Q}} \times \mathbb{G}_{\mathrm{m},\mathbb{Q}} \to \mathbb{P}^3_\mathbb{Q}$ given by

$$\iota(t_1, t_2) = (1 : 2 : t_1 : t_2).$$

As in the examples in the previous section, we denote by X the normalization of the closure of the image of ι and $\overline{D} = \iota^*(\overline{H}^{\mathrm{can}})$. In this case, $X = \mathbb{P}^2_{\mathbb{Q}}$ and D is the divisor at infinity.

We have that Δ_D is the standard simplex of $N_{\mathbb{R}} = \mathbb{R}^2$ and $\Psi_D \colon \mathbb{R}^2 \to \mathbb{R}$ is the function given by

$$\Psi_D(u_1, u_2) = \min(0, u_1, u_2).$$

By [BPS14a, Example 4.3.21], the local metric functions are given, for $(u_1, u_2) \in \mathbb{R}^2$, by

$$\psi_{\overline{D},v}(u_1,u_2) = \begin{cases} \Psi_D\big(u_1 + \log(2), u_2 + \log(2)\big) - \log(2) & \text{if } v = \infty, \\ \Psi_D(u_1,u_2) & \text{if } v \neq \infty. \end{cases}$$

By [BPS14a, Example 5.1.16], the local roof functions are given, for $(x_1, x_2) \in \Delta_D$, by

$$\vartheta_{\overline{D},v}(x_1,x_2) = \begin{cases} (1 - x_1 - x_2)\log(2) & \text{if } v = \infty, \\ 0 & \text{if } v \neq \infty. \end{cases}$$

Hence the global roof function agrees with $\vartheta_{\overline{D},\infty}$. Its only maximizing point is $x_{\max} = (0,0)$. Hence $\partial\vartheta_{\overline{D},\infty}(0,0) = (-\log(2), -\log(2)) + \mathbb{R}^2_{\geq 0}$ and $\partial\vartheta_{\overline{D},v}(0,0) = \mathbb{R}^2_{\geq 0}$ for $v \neq \infty$. Thus

$$\begin{aligned} B_\infty &= [-\log(2), 0]^2, \quad & F_\infty &= (-\log(2), -\log(2)) + \mathbb{R}^2_{\geq 0}, \\ B_v &= [0, \log(2)]^2, & F_v &= \mathbb{R}^2_{\geq 0} \ \text{ for } v \neq \infty. \end{aligned}$$

We also have $\upmu^{\mathrm{ess}}_{\overline{D}}(X) = \vartheta(0,0) = \log(2)$.

Let $(z_0 : z_1 : z_2)$ be homogeneous coordinates of X and consider the curve C of equation $z_0 + z_1 + z_2 = 0$. In what follows, we will see that this curve is a $\overline{D}$-special subvariety. Since C is not a translate of a subtorus, this will show that $\overline{D}$ does not satisfy the Bogomolov property.

For $l \geq 1$ choose a primitive l-th root of the unity ω_l. Let $z_{1,l}$ be a solution of the equation $z^2 + z + \omega_l = 0$ and put $z_{2,l} = \omega_l / z_{1,l}$ for the other solution. Then

$$z_{1,l} + z_{2,l} + 1 = 0 \quad \text{ and } \quad z_{1,l} z_{2,l} = \omega_l. \tag{6.2}$$

In particular, $p_l = (1 : z_{1,l} : z_{2,l})$ is an algebraic point of C.

Let $v \in \mathfrak{M}_{\mathbb{Q}}$ and $q = (1 : q_1 : q_2) \in \mathrm{Gal}(p_l)_v$. If $v \neq \infty$, then the conditions (6.2) imply that

$$\mathrm{val}_v(q) = (0,0) \in B_v. \tag{6.3}$$

If $v = \infty$, then these same conditions (6.2) give $\max(|q_1|_\infty, |q_2|_\infty) \leq \frac{1+\sqrt{5}}{2}$. Thus

$$\mathrm{val}_\infty(q) \in \Big(-\log\Big(\frac{1+\sqrt{5}}{2}\Big), -\log\Big(\frac{1+\sqrt{5}}{2}\Big)\Big) + \mathbb{R}^2_{\geq 0} \subset F_\infty. \tag{6.4}$$

Moreover, by the product formula and (6.3), we have

$$\mathrm{E}[\nu_{p_l,\infty}] = \frac{1}{\#\,\mathrm{Gal}(p_l)_\infty} \sum_{q \in \mathrm{Gal}(p_l)_\infty} \mathrm{val}_\infty(q) = (0,0) \in B_\infty. \tag{6.5}$$

By Corollary 4.9, the conditions (6.3), (6.4) and (6.5) imply that $\mathrm{h}_{\overline{D}}(p_l) = \upmu^{\mathrm{ess}}_{\overline{D}}(X)$. Since the sequence $(p_l)_{l \geq 1}$ is generic in C, we deduce $\upmu^{\mathrm{ess}}_{\overline{D}}(C) = \upmu^{\mathrm{ess}}_{\overline{D}}(X)$ and so C is a $\overline{D}$-special subvariety.

We generalize this example to a family of metrics on toric varieties of dimension greater than or equal to 2.

Proposition 6.7. *Let X be a proper toric variety over $\mathbb{Q}$ of dimension $n \geq 2$ and D a big and nef $\mathbb{R}$-divisor on X. Let $u_0 \in N_{\mathbb{R}}$ and consider the metrized divisor $\overline{D}^{u_0}$ over D defined by*

$$\psi_{\overline{D}^{u_0},v}(u) = \begin{cases} \Psi_D(u-u_0) & \text{if } v = \infty, \\ \Psi_D(u) & \text{if } v \neq \infty. \end{cases}$$

Then $\overline{D}^{u_0}$ satisfies the Bogomolov property if and only if $u_0 = 0$.

Proof. When $u_0 = 0$ we have $\overline{D}^{u_0} = \overline{D}^{\mathrm{can}}$. By Theorem 5.12 and Example 5.16, this toric metrized divisor satisfies the Bogomolov property.

Suppose $u_0 \neq 0$. The local roof functions of $\overline{D}^{u_0}$ are given, for $x \in \Delta_D$, by

$$\vartheta_{\overline{D}^{u_0},v}(x) = \begin{cases} \langle x, u_0\rangle & \text{if } v = \infty, \\ 0 & \text{if } v \neq \infty. \end{cases}$$

In particular, the global roof function $\vartheta_{\overline{D}}$ coincides with $\vartheta_{\overline{D}^{u_0},\infty}$. Fix x_0 in the relative interior of the convex subset of Δ_D on which $\vartheta_{\overline{D}}$ attains it maximum value. If we denote by ϑ_0 the constant function equal to 0 defined on Δ_D, then $\sigma_0 = \partial\vartheta_0(x_0)$ is a cone in $N_{\mathbb{R}}$ containing $-u_0$ in its relative interior. Moreover,

$$\partial\vartheta_{\overline{D}^{u_0},\infty}(x_0) = u_0 + \sigma_0 \quad \text{and} \quad \partial\vartheta_{\overline{D}^{u_0},v}(x_0) = \sigma_0 \text{ for } v \neq \infty.$$

It follows that $0 \in B_v$ for every v, that $F_\infty = u_0 + \sigma_0$ and that $F_v = \sigma_0$ for $v \neq \infty$.

As in Example 6.6, to prove that $\overline{D}^{u_0}$ does not satisfy the Bogomolov property, it is enough to exhibit a curve C in X which is $\overline{D}$-special but not a translate of a subtorus.

We identify $N_{\mathbb{R}} \simeq \mathbb{R}^n$. Since X is proper and σ_0 is a cone of the fan of X, there is a primitive vector $n_0 \in N$ in σ_0. It follows that there is $\varepsilon_0 > 0$ such that

$$\ell_0 := \{xn_0 \mid -\varepsilon_0 \leq x \leq \varepsilon_0\} \subset u_0 + \sigma_0.$$

Choose a primitive vector $a_0 \in N$ such that a_0 and n_0 generate a saturated sublattice V of N. Put $b_0 = n_0 + a_0$. Then a_0 and b_0 form an integral basis of V. Fix an integer $k_0 \geq \varepsilon_0^{-1}$ and consider the linear map $L\colon V_{\mathbb{R}} \to \mathbb{R}^2$ defined by

$$L(sa_0 + tb_0) = k_0 \cdot (s,t).$$

Let S be the toric surface in X_0 associated to the saturated sublattice V. The linear map L induces a toric morphism $\iota\colon S \to \mathbb{G}^2_{\mathrm{m},\mathbb{Q}}$. Let C be the curve in $\mathbb{G}^2_{\mathrm{m},\mathbb{Q}}$ of equation $x + y + 1 = 0$ and denote by C_0 the closure in X of the curve $\iota^{-1}(C)$.

As in Example 6.6, for $l \geq 1$ choose a primitive l-th root of unity root ω_l. Let $z_{1,l}$ be a solution of the equation $z^2 + z + \omega_l = 0$ and put $z_{2,l} = \omega_l/z_{1,l}$. Hence

$$z_{1,l} + z_{2,l} + 1 = 0 \quad \text{and} \quad z_{1,l}z_{2,l} = \omega_l.$$

In particular, $(z_{1,l}, z_{2,l}) \in C(\overline{\mathbb{Q}})$. Choose a point $p_l \in C_0(\overline{\mathbb{Q}})$ such that $\iota(p_l) = (z_{1,l}, z_{2,l})$. The sequence of points $(p_l)_{l\geq 0}$ is generic in C_0.

For every place v there is a commutative diagram

$$\begin{tikzcd} (\mathbb{G}^2_{\mathrm{m}})^{\mathrm{an}}_v \arrow[d, "\mathrm{val}_v"] & S^{\mathrm{an}}_v \arrow[l, "\iota"'] \arrow[r, hook] \arrow[d, "\mathrm{val}_v"] & X^{\mathrm{an}}_{0,v} \arrow[d, "\mathrm{val}_v"] \\ \mathbb{R}^2 & V_{\mathbb{R}} \arrow[l, "L"'] \arrow[r, hook] & N_{\mathbb{R}} \end{tikzcd}$$

Since $n_0 = b_0 - a_0$, we have

$$\ell := L_{\mathbb{R}}(\ell_0) = \{(x,-x) \mid |x| \leq \varepsilon_0 k_0\}.$$

Arguing as in Example 6.6, for every non-Archimedean place v and every point $q \in \mathrm{Gal}(p_l)_v$, we have

$$\mathrm{val}_v(\iota(q)) = 0.$$

Since L is injective, $\mathrm{val}_v(q) = 0$ and therefore $\nu_{p_l,v} = \delta_0$. In particular,

$$\mathrm{supp}(\nu_{p_l,v}) = \{0\} \subset F_v \text{ and } \mathrm{E}[\nu_{p_l,v}] = 0 \in B_v.$$

When $v = \infty$, the product formula implies that

$$\mathrm{E}[\nu_{p_l,\infty}] = \frac{1}{\# \mathrm{Gal}(p_l)_\infty} \sum_{q \in \mathrm{Gal}(p_l)_\infty} \mathrm{val}_\infty(q) = 0 \in B_\infty.$$

On the other hand, the facts that $|z_{1,l}|_\infty \, |z_{1,l}|_\infty = 1$ and that

$$\frac{\sqrt{5}-1}{2} \leq \min\{|z_{1,l}|_\infty, |z_{2,l}|_\infty\} \leq \max\{|z_{1,l}|_\infty, |z_{2,l}|_\infty\} \leq \frac{1+\sqrt{5}}{2},$$

imply that $\mathrm{val}_\infty(\iota(q)) \in \ell$ for every $q \in \mathrm{Gal}(p_l)_\infty$. Thus

$$\mathrm{val}_\infty(q) \in \ell_0 \subset u_0 + \sigma_0 = F_\infty.$$

This implies that $\mathrm{supp}(\nu_{p_l,\infty}) \subset F_\infty$. By Lemma 4.8, we have $\mathrm{h}_{\overline{D}}(p_l) = \upmu^{\mathrm{ess}}_{\overline{D}}(X)$. Being the sequence $(p_l)_{l\geq 1}$ generic in C_0, we deduce that C_0 is $\overline{D}$-special. Since C_0 is not a translate of a subtorus, we conclude that $\overline{D}$ does not satisfy the Bogomolov property, as stated. □

7. Potential theory on the projective line and small points

In this section, we apply potential theory on the projective line over a number field, and in particular Rumely's Fekete-Szegő theorem, to produce interesting sequences of small points in the non-monocritical case.

In the absence of modulus concentration, this allows to produce a wealth of non-toric measures that are limit measures of Galois orbits of generic sequences of points of small height. These techniques also allow to show that the absence of modulus concentration at a place can affect the equidistribution property at another place.

7.1. **Limit measures in the absence of modulus concentration.** We recall the basic objects of potential theory on the projective line. For most of the details and precise definitions, we refer the reader to [Tsu75] and [BR10] for the Archimedean and non-Archimedean cases, respectively.

Let $\mathbb{K}$ be a number field and fix a place $v \in \mathfrak{M}_{\mathbb{K}}$. For a subset $E \subset \mathbb{C}_v$, we denote by $\overline{E}$ its closure in $\mathbb{A}^{1,\mathrm{an}}_v$. Moreover, for $r > 0$, put

$$\mathcal{B}_v(E, r) = \Big\{ z \in \mathbb{C}_v \ \Big| \ \inf_{a \in E} |z - a|_v \leq r \Big\}.$$

In particular, for $a \in \mathbb{C}_v$ the set $\mathcal{B}_v(a, r)$ is the closed ball with center a and radius r. We set $O_v = \mathcal{B}_v(0, 1)$.

We denote by

$$\delta_v \colon \mathbb{A}^{1,\mathrm{an}}_v \times \mathbb{A}^{1,\mathrm{an}}_v \to \mathbb{R}$$

the function defined by $\delta_v(z, z') = |z - z'|_v$ for v Archimedean, and as the unique upper semicontinuous extension of the function on $\mathbb{C}_v \times \mathbb{C}_v$ defined by $(z, z') \mapsto |z - z'|_v$ for v non-Archimedean, see [BR10, Proposition 4.1].

Given a probability measure μ on $\mathbb{A}^{1,\mathrm{an}}_v$, the *energy integral* (with respect to the point at infinity) of μ is defined as

$$I_v(\mu) = \int_{\mathbb{A}^{1,\mathrm{an}}_v \times \mathbb{A}^{1,\mathrm{an}}_v} -\log(\delta_v(z, z')) \, \mathrm{d}\mu(z) \, \mathrm{d}\mu(z'). \tag{7.1}$$

Let $K \subset \mathbb{A}_v^{1,\mathrm{an}}$ be a measurable subset. The *v-adic Robin constant* and *capacity* (with respect to the point at infinity) of K are respectively defined as

$$V_v(K) = \inf\{I_v(\mu) \mid \mathrm{supp}(\mu) \subset K\} \quad \text{and} \quad \mathrm{cap}_v(K) = \mathrm{e}^{-V_v(K)}. \tag{7.2}$$

In the non-Archimedean case, $\mathbb{C}_v$ is a proper subset of $\mathbb{A}_v^{1,\mathrm{an}}$. In general,

$$\mathrm{cap}_v(K \cap \mathbb{C}_v) \leq \mathrm{cap}_v(K),$$

but there is a particular case where the equality holds. If K is a closed strict affinoid domain, then $K \cap \mathbb{C}_v$ is an RL-domain in the sense of [Rum89]. Moreover, $\overline{K \cap \mathbb{C}_v} = K$ and, by [BR10, Corollary 6.26] and [Rum89, Theorem 4.3.3], we have

$$\mathrm{cap}_v(K \cap \mathbb{C}_v) = \mathrm{cap}_v(K).$$

For instance, O_v is a closed strict affinoid domain and

$$\mathrm{cap}_v(O_v) = \mathrm{cap}_v(\overline{O}_v) = 1. \tag{7.3}$$

In the Archimedean case, $O_v = \overline{O}_v$ and these equalities are also valid.

If K is compact and $\mathrm{cap}_v(K) > 0$, then there exists a unique probability measure, denoted by ρ_K, supported on K and realizing the infimum in (7.2), see [Tsu75, §III.2 and Theorem III.32] for the Archimedean case and [BR10, Propositions 6.6 and 7.21] for the non-Archimedean one. Hence

$$I_v(\rho_K) = V_v(K).$$

This measure is called the *equilibrium measure* of K. For $K = \overline{O}_v$, it agrees with $\lambda_{\mathbb{S}_v,0}$, the Haar probability measure on the unit circle when v is Archimedean, and the Dirac measure at the Gauss point of $\mathbb{A}_v^{1,\mathrm{an}}$ when v is non-Archimedean.

Definition 7.1. An *adelic set* is a collection $\boldsymbol{E} = (E_v)_{v\in\mathfrak{M}_\mathbb{K}}$ such that E_v is a subset of $\mathbb{C}_v$ invariant under the action of the absolute v-adic Galois group $\mathrm{Gal}(\overline{\mathbb{K}}_v/\mathbb{K}_v)$ for all v, and such that $E_v = O_v$ for all but a finite number of v. We say that $\boldsymbol{E}$ is *bounded* (respectively *closed*, *open*) if E_v is bounded (respectively closed, open) for all v.

Given an adelic set $\boldsymbol{E} = (E_v)_{v\in\mathfrak{M}_\mathbb{K}}$, its *(global) capacity* is defined as

$$\mathrm{cap}(\boldsymbol{E}) = \prod_{v\in\mathfrak{M}_\mathbb{K}} \mathrm{cap}_v(E_v)^{n_v}.$$

By (7.3), this product actually runs over a finite set and so the global capacity is well-defined.

The following result shows that, in the non-monocritical case, there is a wealth of measures not invariant under the action of the compact torus, that can be obtained as limit measures of Galois orbits of generic sequences of points of small height.

Theorem 7.2. *Let $X = \mathbb{P}^1_\mathbb{K}$ and $\overline{D}$ the divisor at infinity equipped with a semi-positive toric metric. Let B_v, F_v be the associated subsets of $N_\mathbb{R} = \mathbb{R}$ as in* (4.4). *Let $\boldsymbol{E} = (E_v)_{v\in\mathfrak{M}_\mathbb{K}}$ be a closed bounded adelic set with E_v an RL-domain for every non-Archimedean place v, and such that* $\mathrm{cap}(\boldsymbol{E}) = 1$. *Assume that the following conditions hold:*

1. $\mathrm{supp}((\mathrm{val}_v)_*\rho_{E_v}) \subset F_v$ *for all* $v \in \mathfrak{M}_\mathbb{K}$;
2. $\mathrm{E}[(\mathrm{val}_v)_*\rho_{E_v}] \in B_v$ *for all* $v \in \mathfrak{M}_\mathbb{K}$;
3. $\sum_{v\in\mathfrak{M}_\mathbb{K}} n_v\, \mathrm{E}[(\mathrm{val}_v)_*\rho_{E_v}] = 0$.

Then there is a generic $\overline{D}$-small sequence $(p_l)_{l\geq 1}$ of algebraic points of $X_0 = \mathbb{G}_{\mathrm{m},\mathbb{K}}$ such that, for every $v \in \mathfrak{M}_\mathbb{K}$, the sequence of probability measures $(\mu_{p_l,v})_{l\geq1}$ converges to ρ_{E_v}.

The proof of this theorem will be given after two preliminary propositions. The next statement is a direct consequence of Rumely's version of the Fekete-Szegő theorem in [Rum02, Theorem 2.1].

Proposition 7.3. *Let $\boldsymbol{E} = (E_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$ be a closed bounded adelic set such that* $\operatorname{cap}(\boldsymbol{E}) \geq 1$. *There exists a generic sequence $(p_l)_{l\geq 1}$ of points of $\overline{\mathbb{K}}^{\times}$ satisfying*

$$\operatorname{Gal}(p_l)_v \subset \mathcal{B}_v\Big(E_v, \frac{1}{l}\Big)$$

for all $l \geq 1$ and $v \in \mathfrak{M}_{\mathbb{K}}$. In particular, $\operatorname{Gal}(p_l)_v \subset E_v$ *for every non-Archimedean place v such that $E_v = O_v$.*

Proof. For $l \geq 1$, consider the bounded adelic neighbourhood $\boldsymbol{U}_l = (U_{l,v})_{v\in\mathfrak{M}_{\mathbb{K}}}$ of $\boldsymbol{E}$ given by

$$U_{l,v} = \mathcal{B}_v\Big(E_v, \frac{1}{l}\Big).$$

By [Rum02, Theorem 2.1], there is an infinite number of points $p \in \overline{\mathbb{K}}^{\times}$ such that $\operatorname{Gal}(p)_v \subset U_{l,v}$ for all v. Inductively, for each $l \geq 1$ we choose p_l as one of these points which is different from $p_{l'}$ for $l' \leq l-1$. □

In the notation of Proposition 7.3, when the adelic set $\boldsymbol{E}$ has capacity 1, the sequence of v-adic Galois orbits of the points p_l equidistribute according to the equilibrium measure of the closure $\overline{E}_v$.

Proposition 7.4. *Let $\boldsymbol{E} = (E_v)_{v\in\mathfrak{M}_{\mathbb{K}}}$ be a closed bounded adelic set with E_v an RL-domain for every non-Archimedean place v and such that* $\operatorname{cap}(\boldsymbol{E}) = 1$. *Let $(p_l)_{l\geq 1}$ be a generic sequence of points of $\overline{\mathbb{K}}^{\times}$ with* $\operatorname{Gal}(p_l)_v \subset \mathcal{B}_v(E_v, \frac{1}{l})$ *for all $l \geq 1$ and $v \in \mathfrak{M}_{\mathbb{K}}$. Then, for all $v \in \mathfrak{M}_{\mathbb{K}}$, the sequence $(\mu_{p_l,v})_{l\geq 1}$ converges to the equilibrium measure of $\overline{E}_v$.*

Proof. By taking a subsequence, we can suppose without loss of generality that, for each v in $\mathfrak{M}_{\mathbb{K}}$, the sequence $(\mu_{p_l,v})_{l\geq 1}$ converges to a probability measure μ_v.

For each $l \geq 1$ and $v \in \mathfrak{M}_{\mathbb{K}}$, put for short $G_{l,v} = \operatorname{Gal}(p_l)_v$ and set

$$d_{l,v} = \frac{1}{\#G_{l,v}(\#G_{l,v}-1)} \sum_{\substack{q,q'\in G_{l,v}\\ q\neq q'}} \log|q-q'|_v.$$

Consider also the probability measure on $\mathbb{P}^{1,\mathrm{an}}_v \times \mathbb{P}^{1,\mathrm{an}}_v$ given by

$$\nu_{l,v} = \frac{1}{\#G_{l,v}(\#G_{l,v}-1)} \sum_{\substack{q,q'\in G_{l,v}\\ q\neq q'}} \delta_q \times \delta_{q'},$$

and note that $(\nu_{l,v})_{l\geq 1}$ converges to $\mu_v \times \mu_v$. The function $\log(\delta_v(\cdot,\cdot))$ is bounded from above on $\mathcal{B}_v(E_v,1) \times \mathcal{B}_v(E_v,1)$. Similarly as in the proof of Lemma 3.8, this property implies that

$$\limsup_{l\to\infty} d_{l,v} = \limsup_{l\to\infty} \int_{\mathbb{P}^{1,\mathrm{an}}_v\times\mathbb{P}^{1,\mathrm{an}}_v} \log(\delta_v(z,z'))\,\mathrm{d}\nu_{l,v}(z,z') \\ \leq -I_v(\mu_v) \leq \log\operatorname{cap}_v(E_v). \quad (7.4)$$

By the product formula, $\sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v d_{l,v} = 0$. Let $S \subset \mathfrak{M}_{\mathbb{K}}$ be a finite set of places containing the Archimedean places and those where $E_v \neq O_v$. In particular,

$d_{l,v} \leq 0$ for $v \notin S$. Hence, for $v \in \mathfrak{M}_{\mathbb{K}}$,

$$\begin{aligned}\liminf_{l\to\infty} d_{l,v} &= \liminf_{l\to\infty} \sum_{w\in\mathfrak{M}_{\mathbb{K}}\setminus\{v\}} -\frac{n_w}{n_v} d_{l,w} \\ &\geq \liminf_{l\to\infty} \sum_{w\in S\setminus\{v\}} -\frac{n_w}{n_v} d_{l,w} \\ &\geq -\sum_{w\in S\setminus\{v\}} \frac{n_w}{n_v} \limsup_{l\to\infty} d_{l,w} \\ &\geq -\sum_{w\in S\setminus\{v\}} \frac{n_w}{n_v} \log(\operatorname{cap}_w(E_w)) \\ &\geq \log(\operatorname{cap}_v(E_v)).\end{aligned}$$

Together with (7.4), this implies $I_v(\mu_v) = -\log \operatorname{cap}_v(E_v)$. It follows that μ_v is the equilibrium measure of E_v, completing the proof. $\square$

Proof of Theorem 7.2. Let $(p_l)_{l\geq 1}$ be a generic sequence of points of $\overline{\mathbb{K}}^\times$ as in Proposition 7.4, that exists thanks to Proposition 7.3. Then Proposition 7.4 implies that, for every $v \in \mathfrak{M}_{\mathbb{K}}$, the sequence of probability measures $(\mu_{p_l,v})_{l\geq 1}$ converges to ρ_{E_v}. Here we have to show that, under the present hypotheses, this sequence of points is $\overline{D}$-small.

Let s_D be the canonical section of $\mathcal{O}(D)$ with $\operatorname{div}(s_D) = D$. This is a global section vanishing only at infinity. Hence its v-adic Green function

$$g_{\overline{D},v} = -\log \|s_D\|_v$$

is a continuous real-valued function on $\mathbb{A}^{1,\mathrm{an}}_v$. Let $S \subset \mathfrak{M}_{\mathbb{K}}$ be a finite set of places containing the Archimedean places, the places where the metric $\|\cdot\|_v$ differs from the canonical one, and those where $E_v \neq O_v$.

By construction, $\operatorname{Gal}(p)_v \subset \mathcal{B}_v(E_v, 1)$ for all v. In particular, for all $v \notin S$ we have $\operatorname{Gal}(p)_v \subset O_v$ and so $g_{\overline{D},v}(q) = 0$ for all $q \in \operatorname{Gal}(p)_v$. Hence

$$\mathrm{h}_{\overline{D}}(p_l) = \sum_{v\in\mathfrak{M}_{\mathbb{K}}} \frac{n_v}{\#\operatorname{Gal}(p_l)_v} \sum_{q\in\operatorname{Gal}(p_l)_v} g_{\overline{D},v}(q) = \sum_{v\in S} n_v \int \widetilde{g}_{\overline{D},v}\, \mathrm{d}\mu_{p_l,v}$$

for any continuous function $\widetilde{g}_{\overline{D},v}$ on $\mathbb{P}^{1,\mathrm{an}}_v$ coinciding with $g_{\overline{D},v}$ on the bounded subset $\mathcal{B}_v(E_v, 1)$.

The measures $\mu_{p_l,v}$ converge to ρ_{E_v} and are supported on the closure $\overline{\mathcal{B}_v(E_v, 1)}$. Also, for all $v \notin S$, we have $\rho_{E_v} = \lambda_{\mathbb{S}_v,0}$ and $g_{\overline{D},v}$ vanishes on the support of this measure. Hence

$$\lim_{l\to\infty} \mathrm{h}_{\overline{D}}(p_l) = \sum_{v\in S} n_v \int \widetilde{g}_{\overline{D},v}\, \mathrm{d}\rho_{E_v} = \sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \int g_{\overline{D},v}\, \mathrm{d}\rho_{E_v}. \tag{7.5}$$

By the condition (3) and the fact that $\boldsymbol{E}$ is an adelic set, we deduce that the collection $\boldsymbol{\nu} = ((\operatorname{val}_v)_*\rho_{E_v})_{v\in\mathfrak{M}_{\mathbb{K}}}$ is a centered adelic measure (Definition 4.4). Moreover, $g_{\overline{D},v} = -\psi_{\overline{D},v} \circ \operatorname{val}_v$ on $\mathbb{A}^{1,\mathrm{an}}_v \setminus \{0\}$. By (7.5) and the fact that the equilibrium measure does not charge any individual point, we have

$$\lim_{l\to\infty} \mathrm{h}_{\overline{D}}(p_l) = -\sum_{v\in\mathfrak{M}_{\mathbb{K}}} n_v \int \psi_{\overline{D},v}\, \mathrm{d}(\operatorname{val}_v)_*\rho_{E_v} = \eta_{\overline{D}}(\boldsymbol{\nu}).$$

Lemma 4.8 together with the conditions (1) and (2) implies that $\eta_{\overline{D}}(\boldsymbol{\nu}) = \upmu^{\mathrm{ess}}_{\overline{D}}(X)$. Hence the sequence $(p_l)_{l\geq 1}$ is $\overline{D}$-small, as stated, finishing the proof of the theorem. $\square$

7.2. **Local modulus concentration and equidistribution.** Corollary 4.13 gives a criterion for a semipositive toric metrized $\mathbb{R}$-divisor to satisfy the modulus concentration property at a given place. Applying it, one can immediately give examples where modulus concentration fails at that place. If this happens, then the equidistribution property also fails at that place.

Can this absence of modulus concentration affect the equidistribution property at another place? The next result on the projective line over a number field shows that this can be the case under a rationality hypothesis.

Proposition 7.5. *Let $X = \mathbb{P}^1_{\mathbb{K}}$ be the projective line over a number field $\mathbb{K}$, $\overline{D}$ the divisor at infinity equipped with a semipositive toric metric, and $v_0 \in \mathfrak{M}_{\mathbb{K}}$. For each $v \in \mathfrak{M}_{\mathbb{K}}$, let B_v be the set introduced in Notation 4.2. Assume that there is a point $p \in X_0(\overline{\mathbb{K}}) = \overline{\mathbb{K}}^{\times}$ such that $\mathrm{val}_v(p) \in B_v$ for all $v \in \mathfrak{M}_{\mathbb{K}}$ and $\mathrm{val}_{v_0}(p) \in \mathrm{ri}(B_{v_0})$.*

If $\overline{D}$ does not satisfy the modulus concentration property at v_0, then $\overline{D}$ does not satisfy the equidistribution property at any place of $\mathbb{K}$.

Proof. Assume that $\overline{D}$ does not satisfy the modulus concentration property at v_0. Let $v \in \mathfrak{M}_{\mathbb{K}}$. If $v = v_0$ then clearly $\overline{D}$ does not satisfy the equidistribution property at v, so we can suppose that $v \neq v_0$. Extending scalars to a suitable large number field and translating by the point p, we can also reduce to the case when $0 \in \mathrm{ri}(B_{v_0})$ and $0 \in B_w$ for all $w \in \mathfrak{M}_{\mathbb{K}}$.

Let F_{v_0} and A_{v_0} be the convex sets given in Notation 4.2. By Corollary 4.13, the set F_{v_0} is not a single point. Since $0 \in \mathrm{ri}(B_{v_0})$ and F_{v_0} is the minimal face of A_{v_0} containing B_{v_0}, there is $\delta > 0$ such that the set F_{v_0} contains the interval $[-\delta, \delta]$. Set

$$c = \frac{\mathrm{e}^{\delta} + \mathrm{e}^{-\delta}}{2} > 1$$

and consider the closed bounded adelic set $\boldsymbol{E} = (E_w)_{w \in \mathfrak{M}_{\mathbb{K}}}$ given by

$$E_{v_0} = \begin{cases} [-2c, 2c] & \text{if } v_0 \text{ is Archimedean,} \\ \mathcal{B}_{v_0}(2, c) & \text{if } v_0 \text{ is non-Archimedean,} \end{cases}$$

$$E_v = \begin{cases} [-2/c, 2/c] & \text{if } v \text{ is Archimedean,} \\ \mathcal{B}_v(2, 1/c) & \text{if } v \text{ is non-Archimedean,} \end{cases}$$

and, for $w \neq v_0, v$,

$$E_w = \begin{cases} [-2, 2] & \text{if } w \text{ is Archimedean,} \\ O_w = \mathcal{B}_w(0, 1) & \text{if } w \text{ is non-Archimedean.} \end{cases}$$

The local capacities of these sets are

$$\mathrm{cap}_{v_0}(E_{v_0}) = c, \quad \mathrm{cap}_v(E_v) = 1/c \quad \text{and} \quad \mathrm{cap}_w(E_w) = 1 \quad \text{for } w \neq v_0, v,$$

see for instance [Rum02, §3]. Hence, the global capacity of $\boldsymbol{E}$ is 1.

Consider the map $R\colon \mathbb{P}^1_{\mathbb{K}} \to \mathbb{P}^1_{\mathbb{K}}$ given by $R(z) = z + \frac{1}{z}$. Using the expression $R(z) - 2 = \frac{(z-1)^2}{z}$, one checks that, for w non-Archimedean,

$$R^{-1}(E_w) \subset \begin{cases} \{z \in \mathbb{C}_{v_0} \mid |z-1|_{v_0}^2 \leq c|z|_{v_0}\} & \text{if } w = v_0, \\ \{z \in \mathbb{C}_v \mid |z-1|_v^2 \leq c^{-1}|z|_v\} & \text{if } w = v, \\ \{z \in \mathbb{C}_w \mid |z^2+1|_w \leq |z|_w\} & \text{if } w \neq v_0, v. \end{cases}$$

Using that $z = \frac{1}{2}(R(z) \pm \sqrt{R(z)^2 - 4})$, one also checks that, for w Archimedean,

$$R^{-1}(E_w) \subset \begin{cases} \{z \in \mathbb{C}_{v_0} \mid c - \sqrt{c^2-1} \leq |z|_{v_0} \leq c + \sqrt{c^2-1}\} & \text{if } w = v_0, \\ \{z \in \mathbb{C}_w \mid |z|_w = 1\} & \text{if } w \neq v_0. \end{cases}$$

Note that $c-\sqrt{c^2-1}=\mathrm{e}^{-\delta}$ and $c+\sqrt{c^2-1}=\mathrm{e}^{\delta}$. We deduce from the previous analysis that, regardless whether v_0 or w are Archimedean or not, we have

$$R^{-1}(E_{v_0})\subset \mathrm{val}_{v_0}^{-1}([-\delta,\delta]) \quad \text{ and } \quad R^{-1}(E_w)\subset \mathrm{val}_w^{-1}(0) \text{ for } w\neq v_0.$$

We represent in Figure 4 the inverse images by R of the sets E_{v_0}, E_v and E_w in the Archimedean case. The point x therein is $x=c^{-1}+i\sqrt{1-c^{-2}}$.

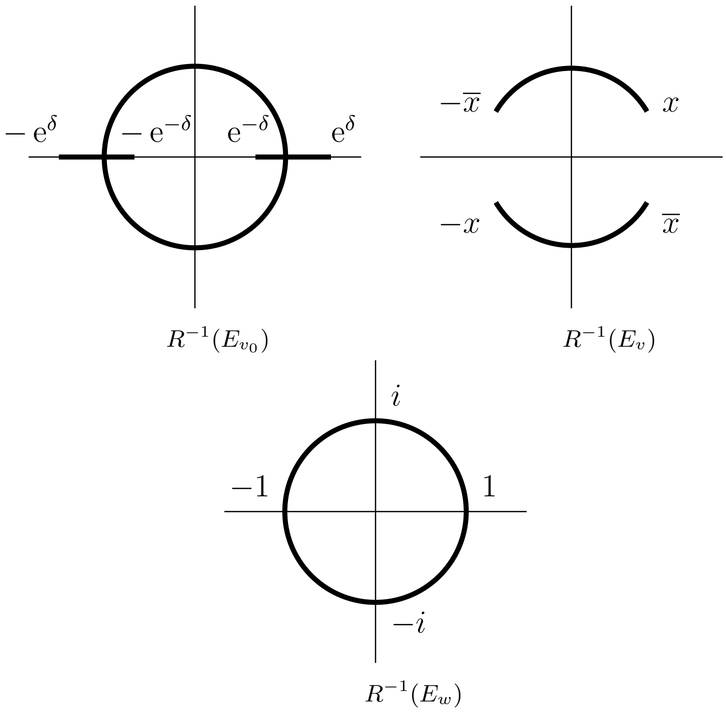

FIGURE 4. Inverse images by R of the sets E_{v_0}, E_v and E_w for v_0, v and $w\neq v,v_0$ Archimedean

Let $(p_l)_{l\geq 1}$ be a generic sequence as given by Proposition 7.3 applied to the adelic set $\boldsymbol{E}$. For each $l\geq 1$, choose a point $q_l\in R^{-1}(p_l)$. After restricting to a subsequence, we can assume that the sequence $(\mu_{q_l,w})_{l\geq 1}$ converges to a probability measure μ_w on $\mathbb{P}^{1,\mathrm{an}}_v$, for all $w\in\mathfrak{M}_{\mathbb{K}}$. By construction, for each w the supports of the direct image measures $\nu_{q_l,w}=(\mathrm{val}_w)_*\mu_{q_l,w}$, $l\geq 1$, are contained in a fixed bounded subset of $N_{\mathbb{R}}=\mathbb{R}$. Therefore, this sequence of measures converges in the KR-topology to the direct image $(\mathrm{val}_w)_*\mu_w$.

Let $S\subset\mathfrak{M}_{\mathbb{K}}$ be the finite subset consisting of the Archimedean places plus v_0 and v. If $w\neq v_0$, then $\mathrm{Gal}(q_l)_w\subset \mathrm{val}_w^{-1}(0)$ and $\mathrm{E}[\nu_{q_l,w}]=0$. Thus

$$\mathrm{E}[(\mathrm{val}_w)_*(\mu_w)]=\lim_l \mathrm{E}[\nu_{q_l,w}]=0.$$

Hence, thanks to the convergence in the KR-topology and the product formula,

$$\mathrm{E}[(\mathrm{val}_{v_0})_*(\mu_{v_0})]=\lim_l \mathrm{E}[\nu_{q_l,v_0}]=\lim_l \sum_{\substack{w\in S\\ w\neq v_0}} -\mathrm{E}[\nu_{q_l,v_0}]=0.$$

Thus $\mathrm{E}[(\mathrm{val}_w)_*(\mu_w)]=0\in B_w$ for all $w\in\mathfrak{M}_{\mathbb{K}}$. By construction, it is also clear that $\mathrm{supp}((\mathrm{val}_w)_*\mu_w)\subset F_w$ for all w. By Lemma 4.8, the sequence $(q_l)_{l\geq 1}$ is $\overline{D}$-small.

We have thus constructed a generic $\overline{D}$-small sequence such that its v-adic Galois orbit converges to a measure μ_v whose support is contained in the closure $\overline{R^{-1}(E_v)}$.

Observe that
$$\overline{R^{-1}(E_v)} \subsetneq \mathbb{S}_v = \mathrm{val}^{-1}(0)$$
because, in the Archimedean case, it does not contain the points 1 and -1, whereas in the non-Archimedean case, it does not contain the Gauss point.

On the other hand, the sequence $(\omega_l)_{l\geq 1}$ given by the choice of a primitive l-th root of unity is also $\overline{D}$-small, but its v-adic Galois orbit converges to the measure $\lambda_{\mathbb{S},0}$. The support of this measure is strictly bigger than that of μ_v. We deduce that $\overline{D}$ does not satisfy the v-adic equidistribution property, as stated. □

Example 7.6. Let $X = \mathbb{P}^1_{\mathbb{Q}}$ and $\overline{D}$ the divisor at infinity plus the divisor at zero, equipped with the semipositive toric metric from Example 6.2. As explained therein, $\overline{D}$ does not satisfy modulus concentration at the place $v_0 = 2$ and, by (6.1), we have $0 \in \mathrm{ri}(B_v)$ for all $v \in \mathfrak{M}_{\mathbb{Q}}$. Theorem 7.2 implies that $\overline{D}$ does not satisfy the equidistribution property for any place of $\mathbb{Q}$.

Remark 7.7. A rationality hypothesis like the condition that the sets B_v contain the image by the valuations map of an algebraic point, is necessary for the conclusion of Proposition 7.5 to hold. Indeed, suppose that, for a given non-Archimedean place v, we have $B_v = F_v = \{u_v\}$ with $u_v \notin \mathrm{val}_v(\overline{\mathbb{K}}_v^{\times})$. By the tree structure of the Berkovich projective line, this implies that $\mathrm{val}_v^{-1}(u_v)$ consists of single point, of type III in Berkovich's classification [BR10, §1.4]. Hence, the v-adic modulus concentration at v given by Corollary 4.13, easily implies that the v-adic Galois orbits of $\overline{D}$-small sequences of algebraic points concentrate around this point of type III, regardless of the structure of the set B_{v_0}.

References

[AW45] E. Artin and G. Whaples, *Axiomatic characterization of fields by the product formula for valuations*, Bull. Amer. Math. Soc. **51** (1945), 469–492.

[BB10] R. Berman and S. Boucksom, *Growth of balls of holomorphic sections and energy at equilibrium*, Invent. Math. **181** (2010), 337–394.

[Ber90] V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Math. Surveys Monogr., vol. 33, Amer. Math. Soc., 1990.

[Bil97] Y. Bilu, *Limit distribution of small points on algebraic tori*, Duke Math. J. **89** (1997), 465–476.

[BMPS12] J. I. Burgos Gil, A. Moriwaki, P. Philippon, and M. Sombra, *Arithmetic positivity on toric varieties*, e-print arXiv:1210.7692v1, 2012, to appear in J. Alg. Geom.

[Bou85] N. Bourbaki, *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations*, Actualités Scientifiques et Industrielles, No. 1308 Hermann, 1964.

[Bou69] ______, *Éléments de mathématique. Fasc. XXXV. Livre VI: Intégration. Chapitre IX: Intégration sur les espaces topologiques séparés*, Actualités Scientifiques et Industrielles, No. 1343, Hermann, 1969.

[BPS14a] J. I. Burgos Gil, P. Philippon, and M. Sombra, *Arithmetic geometry of toric varieties. Metrics, measures and heights*, Astérisque, vol. 360, Soc. Math. France, 2014.

[BPS14b] ______, *Successive minima of toric height functions*, e-print arXiv:1403.4048, 2014, to appear in Ann. Inst. Fourier (Grenoble).

[BR06] M. Baker and R. Rumely, *Equidistribution of small points, rational dynamics, and potential theory*, Ann. Inst. Fourier (Grenoble) **56** (2006), 625–688.

[BR10] ______, *Potential theory and dynamics on the Berkovich projective line*, Math. Surveys Monogr., vol. 159, Amer. Math. Soc., 2010.

[Cha00] A. Chambert-Loir, *Points de petite hauteur sur les variétés semi-abéliennes*, Ann. Sci. Éc. Norm. Sup. **33** (2000), 789–821.

[Cha06] ______, *Mesures et équidistribution sur les espaces de Berkovich*, J. Reine Angew. Math. **595** (2006), 215–235.

[Che11] H. Chen, *Differentiability of the arithmetic volume function*, J. Lond. Math. Soc. (2) **84** (2011), 365–384.

[Cin11] Z. Cinkir, *Zhang's conjecture and the effective Bogomolov conjecture over function fields*, Invent. Math. **183** (2011), 517–562.

[FR06] C. Favre and J. Rivera-Letelier, *Équidistribution quantitative des points de petite hauteur sur la droite projective*, Math. Ann. **335** (2006), 311–361, *corrigendum* in Math. Ann. **339** (2007), 799–801.

[Ghi09] D. Ghioca, *Points of small height on varieties defined over a function field*, Canad. Math. Bull. **52** (2009), 237–244.

[Gub07] W. Gubler, *The Bogomolov conjecture for totally degenerate abelian varieties*, Invent. Math. **169** (2007), 377–400.

[Gub08] ———, *Equidistribution over function fields*, Manuscripta Math. **127** (2008), 485–510.

[Liu02] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxf. Grad. Texts Math., vol. 6, Oxford Univ. Press, 2002.

[Rum89] R. Rumely, *Capacity theory on algebraic curves*, Lecture Notes in Math., vol. 1378, Springer-Verlag, 1989.

[Rum02] ———, *The Fekete-Szegő theorem with splitting conditions. II*, Acta Arith. **103** (2002), 347–410.

[SUZ97] L. Szpiro, E. Ullmo, and S.-W. Zhang, *Équirépartition des petits points*, Invent. Math. **127** (1997), 337–347.

[Tsu75] M. Tsuji, *Potential theory in modern function theory*, Chelsea Publishing Co., 1975.

[Ull98] E. Ullmo, *Positivité et discrétion des points algébriques des courbes*, Annals of Math. **147** (1998), 167–179.

[Vil09] C. Villani, *Optimal transport. Old and new*, Grundlehren Math. Wiss., vol. 338, Springer-Verlag, 2009.

[Wei74] A. Weil, *Basic number theory*, third ed., Grundlehren Math. Wiss., vol. 144, Springer-Verlag, 1974.

[Yam13] K. Yamaki, *Geometric Bogomolov conjecture for abelian varieties and some results for those with some degeneration (with an appendix by Walter Gubler: the minimal dimension of a canonical measure)*, Manuscripta Math. **142** (2013), 273–306.

[Yua08] X. Yuan, *Big line bundles over arithmetic varieties*, Invent. Math. **173** (2008), 603–649.

[Zha95] S.-W. Zhang, *Positive line bundles on arithmetic varieties*, J. Amer. Math. Soc. **8** (1995), 187–221.

[Zha98] ———, *Equidistribution of small points of abelian varieties*, Annals of Math. **147** (1998), 159–165.

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